Lógica y Metodos Avanzados de Razonamiento

Today: Natural Deduction
A logical proof calculus

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Overview:

- Last time:
  - Propositional logics (Syntax, Semantics)
  - First-order Logics (Syntax, Semantics)
- Today:
  - Propositional natural deduction
    - Soundness, completeness
  - First-order natural deduction
  - Examples

Natural Deduction

calculus for reasoning about propositions

DEFINITION

➜ sequent

$$\phi_1, \phi_2, \ldots, \phi_n \quad \vdash \quad \psi$$

with $\phi_1, \phi_2, \ldots, \phi_n, \psi$ propositional formulas

Natural Deduction

calculus for reasoning about propositions

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➜ sequent

$$\underbrace{\phi_1, \phi_2, \ldots, \phi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}}$$

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➜ sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if ψ can be proved from premises $\phi_1, \phi_2, \ldots, \phi_n$ using proof rules of natural deduction

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➜ sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if ψ can be proved from premises $\phi_1, \phi_2, \ldots, \phi_n$ using proof rules of natural deduction

➜ natural deduction consists of 12 basic proof rules and 4 derived rules

➩ and introduction

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$$

- and introduction

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge\mathrm{i}$$

$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

- and introduction

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$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

1	p	premise

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$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

1	p	premise
2	q	premise

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3	r	premise

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2	q	premise
3	r	premise
4	$r \wedge q$	$\wedge$i 3, 2

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$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

1	p	premise
2	q	premise
3	r	premise
4	$r \wedge q$	$\wedge$i 3, 2
5	$(r \wedge q) \wedge p$	$\wedge$i 4, 1

➩ and elimination

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

- **and elimination**

$$\frac{\phi \wedge \psi}{\phi}\ \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge e_2$$

$p \wedge q, r \vdash r \wedge q$ is valid:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$r \wedge q$	$\wedge i$ 2, 3

➩ double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction — derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction — derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction — derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise

$\neg\neg p \wedge r$

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction — derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	
5	r	
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
5	r	
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

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$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
4	$q \wedge r$	
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

- double negation elimination

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e}$$

- double negation introduction derived rule

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
4	$q \wedge r$	$\neg\neg$e 2
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

➩ implication elimination modus ponens

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$$

➯ implication elimination modus ponens

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$$

$p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$ is valid:

1	p	premise
2	$p \rightarrow q$	premise
3	$p \rightarrow (q \rightarrow r)$	premise
4	q	$\rightarrow$e 2, 1
5	$q \rightarrow r$	$\rightarrow$e 3, 1
6	r	$\rightarrow$e 5, 4

- modus tollens derived rule

$$\frac{\phi \longrightarrow \psi \quad \neg\psi}{\neg\phi}\ \text{MT}$$

- modus tollens derived rule

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \text{MT}$$

$\neg p \rightarrow q, \neg q \vdash p$ is valid:

1	$\neg p \rightarrow q$	premise
2	$\neg q$	premise
3	$\neg\neg p$	MT 1, 2
4	p	$\neg\neg$e 3

- implication introduction

$$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$$

➩ implication introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \longrightarrow \psi} \longrightarrow\text{i}$$

box — assumption

➩ implication introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\text{i}$$

box — assumption

$\neg q \rightarrow \neg p \vdash p \rightarrow q$ is valid:

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
5	q	
6	$p \rightarrow q$	

- implication introduction

box

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi} \; \rightarrow\text{i}$$

assumption

$\neg q \rightarrow \neg p \vdash p \rightarrow q$ is valid:

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
5	q	
6	$p \rightarrow q$	

- implication introduction

box

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow\text{i}$$

assumption

$\neg q \rightarrow \neg p \vdash p \rightarrow q$ is valid:

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1, 3
5	q	
6	$p \rightarrow q$	

➩ implication introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to\mathrm{i}$$

box: the boxed part; assumption: ϕ

$\neg q \to \neg p \vdash p \to q$ is valid:

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1, 3
5	q	$\neg\neg$e 4
6	$p \to q$	

➯ implication introduction

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow\mathrm{i}$$

box — assumption (ϕ)

$\neg q \rightarrow \neg p \vdash p \rightarrow q$ is valid:

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1, 3
5	q	$\neg\neg$e 4
6	$p \rightarrow q$	$\rightarrow$i 2–5

- or introduction

$$\frac{\phi}{\phi \vee \psi}\ \vee i_1 \qquad \frac{\psi}{\phi \vee \psi}\ \vee i_2$$

- or introduction

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2$$

$p \wedge q \vdash \neg q \vee p$ is valid:

1	$p \wedge q$	premise
2	p	$\wedge e_1$ 1
3	$\neg q \vee p$	$\vee i_2$ 2

- or elimination

$$\frac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi} \vee e$$

➩ or elimination

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \ \vee\mathrm{e}$$

$p \vee q \vdash q \vee p$ is valid:

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption
5	$q \vee p$	$\vee\mathrm{i}_1$ 4
6	$q \vee p$	$\vee\mathrm{e}$ 1, 2–3, 4–5

DEFINITION

theorem is logical formula ϕ such that sequent $\vdash \phi$ is valid

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EXAMPLE

$p \vee q \to q \vee p$ is theorem:

1	$p \vee q$	assumption
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2
4	q	assumption
5	$q \vee p$	$\vee i_1$ 4
6	$q \vee p$	$\vee e$ 1, 2–3, 4–5
7	$p \vee q \to q \vee p$	$\to i$ 1–6

Summary

	introduction	elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1$ $\quad$ $\dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1$ $\quad$ $\dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$

	introduction	elimination
$\neg\neg$	$\dfrac{\phi}{\neg\neg\phi}$ $\neg\neg$i	$\dfrac{\neg\neg\phi}{\phi}$ $\neg\neg$e

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} \text{ MT}$$

TO BE CONTINUED

DEFINITION

➜ contradiction

$$\phi \land \neg\phi \qquad\qquad \neg\phi \land \phi$$

with ϕ propositional formula

DEFINITION

➜ contradiction

$$\phi \wedge \neg\phi \qquad\qquad \neg\phi \wedge \phi$$

with ϕ propositional formula

THEOREM

all contradictions are equivalent:

ϕ and ψ are contradictions $\implies$ $\phi \vdash \psi$ and $\psi \vdash \phi$ are valid

DEFINITION

- contradiction

$$\phi \wedge \neg\phi \qquad\qquad \neg\phi \wedge \phi$$

  with ϕ propositional formula

- new symbol $\perp$ ("bottom") represents all contradictions

THEOREM

all contradictions are equivalent:

$$\phi \text{ and } \psi \text{ are contradictions} \quad \Longrightarrow \quad \phi \vdash \psi \text{ and } \psi \vdash \phi \text{ are valid}$$

- bottom elimination

$$\frac{\bot}{\phi} \; \bot e$$

- bottom elimination

$$\frac{\bot}{\phi}\ \bot e$$

- negation elimination

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

- bottom elimination

$$\frac{\bot}{\phi}\ \bot e$$

- negation elimination

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

- negation introduction

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg i$$

$p \to q, p \to \neg q \vdash \neg p$ is valid: :

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 3–6

$p \rightarrow q, p \rightarrow \neg q \vdash \neg p$ is valid:

1	$p \rightarrow q$	premise
2	$p \rightarrow \neg q$	premise
3	p	assumption
4	q	$\rightarrow$e 1, 3
5	$\neg q$	$\rightarrow$e 2, 3
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 3–6

$p, p \rightarrow q, p \rightarrow \neg q \vdash r$ is valid:

1	p	premise
2	$p \rightarrow q$	premise
3	$p \rightarrow \neg q$	premise
4	q	$\rightarrow$e 2, 1
5	$\neg q$	$\rightarrow$e 3, 1
6	$\bot$	$\neg$e 4, 5
7	r	$\bot$e 6

- proof by contradiction   derived rule

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC}$$

- proof by contradiction derived rule

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \text{PBC}$$

- law of excluded middle derived rule

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

$p \to q \vdash \neg p \lor q$ is valid:

1	$p \to q$	premise
2	$\neg p \lor p$	LEM
3	$\neg p$	assumption
4	$\neg p \lor q$	$\lor i_1$ 3
5	p	assumption
6	q	$\to e$ 1, 5
7	$\neg p \lor q$	$\lor i_2$ 6
8	$\neg p \lor q$	$\lor e$ 2, 3–4, 5–7

Summary

	introduction	elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$		$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

derived proof rules

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \text{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

DEFINITION

formulas ϕ and ψ are provably equivalent ($\phi \dashv\vdash \psi$) if both $\phi \vdash \psi$ and $\psi \vdash \phi$ are valid

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THEOREM

proof rules LEM, PBC and $\neg\neg$e are inter-derivable (wrt basic proof rules)

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proof rules LEM, PBC and $\neg\neg$e are inter-derivable (wrt basic proof rules) and controversial because they are not constructive

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proof rules LEM, PBC and $\neg\neg$e are inter-derivable (wrt basic proof rules) and controversial because they are not constructive

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classical logicians use all proof rules

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THEOREM

proof rules LEM, PBC and $\neg\neg$e are inter-derivable (wrt basic proof rules) and controversial because they are not constructive

DEFINITION

classical logicians use all proof rules

intuitionistic logicians do not use LEM, PBC and $\neg\neg$e

THEOREM

propositional logic is sound:

$$\phi_1, \phi_2, \ldots, \phi_n \ \vdash \ \psi \ \text{ is valid} \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \ \models \ \psi$$

THEOREM

propositional logic is sound:

$$\phi_1, \phi_2, \ldots, \phi_n \ \vdash \ \psi \quad \text{is valid} \qquad \Longrightarrow \qquad \phi_1, \phi_2, \ldots, \phi_n \ \models \ \psi$$

only true statements can be proved

THEOREM

propositional logic is sound:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid} \implies \phi_1, \phi_2, \ldots, \phi_n \models \psi$$

only true statements can be proved

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \implies \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

THEOREM

propositional logic is sound:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid} \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \models \psi$$

only true statements can be proved

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF IDEA (SOUNDNESS)

use induction on length of natural deduction proof
and case analysis of last proof step

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PROBLEM

initial part of proof need not correspond to sequent with same premises

1	$p \rightarrow q$	premise
2	$p \rightarrow \neg q$	premise
3	p	assumption
4	q	$\rightarrow$e 1, 3
5	$\neg q$	$\rightarrow$e 2, 3
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 3–6

PROOF IDEA (SOUNDNESS)

use induction on length of natural deduction proof
and case analysis of last proof step

PROBLEM

initial part of proof need not correspond to sequent with same premises

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	
4	q	$\to$e 1, 3	
5	$\neg q$	$\to$e 2, 3	
6	$\bot$	$\neg$e 4, 5	
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

PROOF IDEA (SOUNDNESS)

use induction on length of natural deduction proof
and case analysis of last proof step

PROBLEM

initial part of proof need not correspond to sequent with same premises

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	?
4	q	$\to$e 1, 3	?
5	$\neg q$	$\to$e 2, 3	?
6	$\bot$	$\neg$e 4, 5	?
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

SOLUTION

add assumptions to sequents

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	
4	q	$\to$e 1, 3	
5	$\neg q$	$\to$e 2, 3	
6	$\bot$	$\neg$e 4, 5	
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

SOLUTION

add assumptions to sequents

1	$p \rightarrow q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow q$
2	$p \rightarrow \neg q$	premise	$p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow \neg q$
3	p	assumption	$p \rightarrow q, p \rightarrow \neg q; p \vdash p$
4	q	$\rightarrow$e 1, 3	$p \rightarrow q, p \rightarrow \neg q; p \vdash q$
5	$\neg q$	$\rightarrow$e 2, 3	$p \rightarrow q, p \rightarrow \neg q; p \vdash \neg q$
6	$\bot$	$\neg$e 4, 5	$p \rightarrow q, p \rightarrow \neg q; p \vdash \bot$
7	$\neg p$	$\neg$i 3–6	$p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

SOLUTION

add assumptions to sequents

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	$p \to q, p \to \neg q; p \vdash p$
4	q	$\to$e 1, 3	$p \to q, p \to \neg q; p \vdash q$
5	$\neg q$	$\to$e 2, 3	$p \to q, p \to \neg q; p \vdash \neg q$
6	$\bot$	$\neg$e 4, 5	$p \to q, p \to \neg q; p \vdash \bot$
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

DEFINITION

extended sequent

$$\underbrace{\phi_1, \phi_2, \ldots, \phi_n}_{\text{premises}};\ \underbrace{\psi_1, \psi_2, \ldots, \psi_m}_{\text{assumptions}} \vdash \underbrace{\chi}_{\text{conclusion}}$$

1	$p \wedge q \rightarrow r$	premise	$p \wedge q \rightarrow r \vdash p \wedge q \rightarrow r$
2	p	assumption	$p \wedge q \rightarrow r;\ p \vdash p$
3	q	assumption	$p \wedge q \rightarrow r;\ p, q \vdash q$
4	$p \wedge q$	$\wedge$i 2, 3	$p \wedge q \rightarrow r;\ p, q \vdash p \wedge q$
5	r	$\rightarrow$e 1, 4	$p \wedge q \rightarrow r;\ p, q \vdash r$
6	$q \rightarrow r$	$\rightarrow$i 3–5	$p \wedge q \rightarrow r;\ p \vdash q \rightarrow r$
7	$p \rightarrow (q \rightarrow r)$	$\rightarrow$i 2–6	$p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

1	$p \wedge q \to r$	premise	$p \wedge q \to r \vdash p \wedge q \to r$
2	p	assumption	$p \wedge q \to r;\ p \vdash p$
3	q	assumption	$p \wedge q \to r;\ p, q \vdash q$
4	$p \wedge q$	$\wedge$i 2, 3	$p \wedge q \to r;\ p, q \vdash p \wedge q$
5	r	$\to$e 1, 4	$p \wedge q \to r;\ p, q \vdash r$
6	$q \to r$	$\to$i 3–5	$p \wedge q \to r;\ p \vdash q \to r$
7	$p \to (q \to r)$	$\to$i 2–6	$p \wedge q \to r \vdash p \to (q \to r)$

SOUNDNESS PROOF

by induction on length of proof of

$$\Phi_1 ; \Phi_2 \ \vdash \ \psi$$

we show that

$$\Phi_1 , \Phi_2 \ \models \ \psi$$

CLAIM

$\Phi_1;\Phi_2 \vdash \psi$ is valid $\Longrightarrow$ $\Phi_1,\Phi_2 \models \psi$

BASE CASES

- ➜ premise $\psi \in \Phi_1 \Longrightarrow \Phi_1,\Phi_2 \models \psi$
- ➜ assumption $\psi \in \Phi_2 \Longrightarrow \Phi_1,\Phi_2 \models \psi$

CLAIM

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\Longrightarrow$ $\Phi_1, \Phi_2 \models \psi$

BASE CASES

- premise $\psi \in \Phi_1 \Longrightarrow \Phi_1, \Phi_2 \models \psi$
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INDUCTION STEP case analysis of last proof step

CLAIM

$\Phi_1 ; \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1 , \Phi_2 \models \psi$

BASE CASES

- premise $\psi \in \Phi_1 \implies \Phi_1 , \Phi_2 \models \psi$
- assumption $\psi \in \Phi_2 \implies \Phi_1 , \Phi_2 \models \psi$

INDUCTION STEP case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

CLAIM

$\Phi_1 ; \Phi_2 \vdash \psi$ is valid $\Longrightarrow$ $\Phi_1 , \Phi_2 \models \psi$

BASE CASES

- premise $\psi \in \Phi_1 \Longrightarrow \Phi_1 , \Phi_2 \models \psi$
- assumption $\psi \in \Phi_2 \Longrightarrow \Phi_1 , \Phi_2 \models \psi$

INDUCTION STEP case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1 ; \Phi_2^1 \vdash \psi_1$ and $\Phi_1 ; \Phi_2^2 \vdash \psi_2$

CLAIM

$\Phi_1 ; \Phi_2 \vdash \psi$ is valid $\Longrightarrow$ $\Phi_1 , \Phi_2 \models \psi$

BASE CASES

- premise $\psi \in \Phi_1 \Longrightarrow \Phi_1 , \Phi_2 \models \psi$
- assumption $\psi \in \Phi_2 \Longrightarrow \Phi_1 , \Phi_2 \models \psi$

INDUCTION STEP case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1 ; \Phi_2^1 \vdash \psi_1$ and $\Phi_1 ; \Phi_2^2 \vdash \psi_2$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

CLAIM

$\Phi_1 ; \Phi_2 \ \vdash \ \psi$ is valid $\implies$ $\Phi_1 , \Phi_2 \ \models \ \psi$

BASE CASES

- premise $\psi \in \Phi_1 \implies \Phi_1 , \Phi_2 \ \models \ \psi$
- assumption $\psi \in \Phi_2 \implies \Phi_1 , \Phi_2 \ \models \ \psi$

INDUCTION STEP case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1 ; \Phi_2^1 \ \vdash \ \psi_1$ and $\Phi_1 ; \Phi_2^2 \ \vdash \ \psi_2$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1 , \Phi_2^1 \ \models \ \psi_1$ and $\Phi_1 , \Phi_2^2 \ \models \ \psi_2$

CLAIM

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1, \Phi_2 \models \psi$

BASE CASES

- premise $\psi \in \Phi_1 \implies \Phi_1, \Phi_2 \models \psi$
- assumption $\psi \in \Phi_2 \implies \Phi_1, \Phi_2 \models \psi$

INDUCTION STEP

case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi_1$ and $\Phi_1; \Phi_2^2 \vdash \psi_2$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi_1$ and $\Phi_1, \Phi_2^2 \models \psi_2$

hence: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2 \implies \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2^1, \Phi_2^2$

CLAIM

$\Phi_1 ; \Phi_2 \ \vdash \ \psi$ is valid $\quad \Longrightarrow \quad \Phi_1 , \Phi_2 \ \models \ \psi$

BASE CASES

- premise $\quad \psi \in \Phi_1 \quad \Longrightarrow \quad \Phi_1 , \Phi_2 \ \models \ \psi$
- assumption $\quad \psi \in \Phi_2 \quad \Longrightarrow \quad \Phi_1 , \Phi_2 \ \models \ \psi$

INDUCTION STEP case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1 ; \Phi_2^1 \ \vdash \ \psi_1$ and $\Phi_1 ; \Phi_2^2 \ \vdash \ \psi_2$ with $\Phi_2^1 , \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1 , \Phi_2^1 \ \models \ \psi_1$ and $\Phi_1 , \Phi_2^2 \ \models \ \psi_2$

hence: $\bar{v}(\phi) = \mathsf{T} \quad \forall \, \phi \in \Phi_1 , \Phi_2 \quad \Longrightarrow \quad \bar{v}(\phi) = \mathsf{T} \quad \forall \, \phi \in \Phi_1 , \Phi_2^1 , \Phi_2^2$

$\Longrightarrow \quad \bar{v}(\psi_1) = \bar{v}(\psi_2) = \mathsf{T}$

CLAIM

$\Phi_1 ; \Phi_2 \ \vdash \ \psi$ is valid $\implies$ $\Phi_1 , \Phi_2 \ \models \ \psi$

BASE CASES

- ➜ premise $\psi \in \Phi_1 \implies \Phi_1 , \Phi_2 \ \models \ \psi$
- ➜ assumption $\psi \in \Phi_2 \implies \Phi_1 , \Phi_2 \ \models \ \psi$

INDUCTION STEP

case analysis of last proof step

$\wedge$i $\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1 ; \Phi_2^1 \ \vdash \ \psi_1$ and $\Phi_1 ; \Phi_2^2 \ \vdash \ \psi_2$ with $\Phi_2^1 , \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1 , \Phi_2^1 \ \models \ \psi_1$ and $\Phi_1 , \Phi_2^2 \ \models \ \psi_2$

hence: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1 , \Phi_2 \implies \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1 , \Phi_2^1 , \Phi_2^2$

$\implies \bar{v}(\psi_1) = \bar{v}(\psi_2) = \mathsf{T}$

$\implies \bar{v}(\psi_1 \wedge \psi_2) = \mathsf{T}$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \ \vdash \ \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \ \models \ \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

$\bar{v}(\psi_1) = \mathsf{F} \quad \Longrightarrow$

$\bar{v}(\psi_1) = \mathsf{T} \quad \Longrightarrow$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

$\bar{v}(\psi_1) = \mathsf{F} \quad \Longrightarrow \quad \bar{v}(\psi_1 \rightarrow \psi_2) = \mathsf{T}$

$\bar{v}(\psi_1) = \mathsf{T} \quad \Longrightarrow$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

$$\bar{v}(\psi_1) = \mathsf{F} \quad \Longrightarrow \quad \bar{v}(\psi_1 \rightarrow \psi_2) = \mathsf{T}$$

$$\bar{v}(\psi_1) = \mathsf{T} \quad \Longrightarrow \quad \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2, \psi_1$$

INDUCTION STEP case analysis of last proof step

→i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

$$\begin{aligned}
\bar{v}(\psi_1) = \mathsf{F} \quad &\Longrightarrow \quad \bar{v}(\psi_1 \rightarrow \psi_2) = \mathsf{T} \\
\bar{v}(\psi_1) = \mathsf{T} \quad &\Longrightarrow \quad \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2, \psi_1 \\
&\Longrightarrow \quad \bar{v}(\psi_2) = \mathsf{T}
\end{aligned}$$

INDUCTION STEP case analysis of last proof step

$\rightarrow$i $\psi = \psi_1 \rightarrow \psi_2$

shorter proof $\Phi_1; \Phi_2, \psi_1 \vdash \psi_2$

induction hypothesis: $\Phi_1, \Phi_2, \psi_1 \models \psi_2$

suppose: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2$

$$\begin{aligned}
\bar{v}(\psi_1) = \mathsf{F} \quad &\Longrightarrow \quad \bar{v}(\psi_1 \rightarrow \psi_2) = \mathsf{T} \\
\bar{v}(\psi_1) = \mathsf{T} \quad &\Longrightarrow \quad \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2, \psi_1 \\
&\Longrightarrow \quad \bar{v}(\psi_2) = \mathsf{T} \\
&\Longrightarrow \quad \bar{v}(\psi_1 \rightarrow \psi_2) = \mathsf{T}
\end{aligned}$$

INDUCTION STEP case analysis of last proof step

$\neg$e $\psi = \bot$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi'$ and $\Phi_1; \Phi_2^2 \vdash \neg\psi'$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

INDUCTION STEP case analysis of last proof step

$\neg e$ $\psi = \bot$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi'$ and $\Phi_1; \Phi_2^2 \vdash \neg\psi'$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi'$ and $\Phi_1, \Phi_2^2 \models \neg\psi'$

hence: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2 \quad \Longrightarrow \quad \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2^1, \Phi_2^2$

$$\Longrightarrow \quad \bar{v}(\psi') = \mathsf{T} \text{ and } \bar{v}(\neg\psi') = \mathsf{T}$$

$$\Longrightarrow \quad \bar{v}(\psi' \wedge \neg\psi') = \mathsf{T}$$

$$\Longrightarrow \quad \bar{v}(\bot) = \mathsf{T}$$

INDUCTION STEP case analysis of last proof step

¬e $\psi = \bot$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi'$ and $\Phi_1; \Phi_2^2 \vdash \neg\psi'$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \models \psi'$ and $\Phi_1, \Phi_2^2 \models \neg\psi'$

hence: $\bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2 \implies \bar{v}(\phi) = \mathsf{T} \quad \forall\, \phi \in \Phi_1, \Phi_2^1, \Phi_2^2$

$$\implies \bar{v}(\psi') = \mathsf{T} \text{ and } \bar{v}(\neg\psi') = \mathsf{T}$$

$$\implies \bar{v}(\psi' \wedge \neg\psi') = \mathsf{T}$$

$$\implies \bar{v}(\bot) = \mathsf{T}$$

hence: $\Phi_1, \Phi_2 \models \bot$

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ easy

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \to (\phi_2 \to (\cdots (\phi_n \to \psi) \cdots))$ easy

② $\vdash \phi_1 \to (\phi_2 \to (\cdots (\phi_n \to \psi) \cdots))$ is valid

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ easy

② $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ is valid difficult

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ easy

② $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ is valid difficult

③ $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

THEOREM

propositional logic is complete:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid}$$

all true statements can be proved

PROOF STRUCTURE

① $\models \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ easy

② $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ is valid difficult

③ $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid easy

① $\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$

① $\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$

PROOF

➜ suppose $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ does not hold

① $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi \qquad \Longrightarrow \qquad \vDash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$

PROOF

➜ suppose $\vDash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ does not hold

➜ $\exists$ valuation v with $\bar{v}(\phi_1) = \cdots = \bar{v}(\phi_n) =$ T and $\bar{v}(\psi) =$ F

① $\phi_1, \phi_2, \ldots, \phi_n \models \psi \quad \Longrightarrow \quad \models \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$

PROOF

- ➜ suppose $\models \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ does not hold
- ➜ $\exists$ valuation v with $\bar{v}(\phi_1) = \cdots = \bar{v}(\phi_n) = \mathsf{T}$ and $\bar{v}(\psi) = \mathsf{F}$
- ➜ $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ does not hold

❑

③ $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots (\phi_n \rightarrow \psi) \cdots))$ is valid $\Longrightarrow$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

③ $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ is valid $\Longrightarrow$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

PROOF

➜ Π: proof of $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$

③ $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ is valid $\Longrightarrow$ $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

PROOF

➜ Π: proof of $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$

➜ proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$:

ϕ_1	premise
$\vdots$	$\vdots$
ϕ_n	premise

③ $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$ is valid $\Longrightarrow$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

PROOF

➜ Π: proof of $\vdash \phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$

➜ proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$:

ϕ_1	premise
$\vdots$	$\vdots$
ϕ_n	premise
Π	
$\phi_1 \to (\phi_2 \to (\cdots(\phi_n \to \psi)\cdots))$	

③ $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots (\phi_n \rightarrow \psi) \cdots))$ is valid $\Longrightarrow$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

PROOF

➜ Π: proof of $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots (\phi_n \rightarrow \psi) \cdots))$

➜ proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$:

ϕ_1	premise
$\vdots$	$\vdots$
ϕ_n	premise
Π	
$\phi_1 \rightarrow (\phi_2 \rightarrow (\cdots (\phi_n \rightarrow \psi) \cdots))$	
$\phi_2 \rightarrow (\cdots (\phi_n \rightarrow \psi) \cdots)$	$\rightarrow$e

③ $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$ is valid $\Longrightarrow$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

PROOF

➜ Π: proof of $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$

➜ proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$:

ϕ_1	premise
$\vdots$	$\vdots$
ϕ_n	premise
Π	
$\phi_1 \rightarrow (\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots))$	
$\phi_2 \rightarrow (\cdots(\phi_n \rightarrow \psi)\cdots)$	$\rightarrow$e
$\vdots$	$\vdots$
ψ	$\rightarrow$e

□

② $\models \phi \quad \Longrightarrow \quad \vdash \phi$

② $\models \phi \implies \vdash \phi$

DEFINITION

valuation v, formula ϕ

$$\langle\phi\rangle^v = \begin{cases} \phi & \text{if } \bar{v}(\phi) = \mathsf{T} \\ \neg\phi & \text{if } \bar{v}(\phi) = \mathsf{F} \end{cases}$$

② $\models \phi \quad \Longrightarrow \quad \vdash \phi$

DEFINITION

valuation v, formula ϕ

$$\langle\phi\rangle^v = \begin{cases} \phi & \text{if } \bar{v}(\phi) = \mathsf{T} \\ \neg\phi & \text{if } \bar{v}(\phi) = \mathsf{F} \end{cases}$$

MAIN LEMMA

$\forall$ formula $\phi \quad \forall$ valuation v

$p_1, \ldots, p_n$ are all atoms in $\phi \quad \Longrightarrow \quad \langle p_1\rangle^v, \ldots, \langle p_n\rangle^v \vdash \langle\phi\rangle^v$ is valid

every line in truth table corresponds to valid sequent

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

➜ $v(p) =$ T: $\langle p \rangle^v = \langle \phi \rangle^v = p$ $p \vdash p$ is valid

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

- ➜ $v(p) =$ T: $\langle p \rangle^v = \langle \phi \rangle^v = p$ $\quad p \vdash p$ is valid
- ➜ $v(p) =$ F: $\langle p \rangle^v = \langle \phi \rangle^v = \neg p$ $\quad \neg p \vdash \neg p$ is valid

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

- $v(p) = \mathsf{T}$: $\quad \langle p \rangle^v = \langle \phi \rangle^v = p \qquad p \vdash p$ is valid
- $v(p) = \mathsf{F}$: $\quad \langle p \rangle^v = \langle \phi \rangle^v = \neg p \qquad \neg p \vdash \neg p$ is valid

INDUCTION STEP 4 cases

$\phi = \neg\psi$

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

➜ $v(p) =$ T: $\langle p \rangle^v = \langle \phi \rangle^v = p$ $p \vdash p$ is valid

➜ $v(p) =$ F: $\langle p \rangle^v = \langle \phi \rangle^v = \neg p$ $\neg p \vdash \neg p$ is valid

INDUCTION STEP 4 cases

$\phi = \neg \psi$

induction hypothesis: $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

PROOF

induction on the structure of ϕ

BASE CASE

$\phi = p$

➜ $v(p) =$ T: $\langle p \rangle^v = \langle \phi \rangle^v = p$ $p \vdash p$ is valid

➜ $v(p) =$ F: $\langle p \rangle^v = \langle \phi \rangle^v = \neg p$ $\neg p \vdash \neg p$ is valid

INDUCTION STEP 4 cases

$\phi = \neg \psi$

induction hypothesis: $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

➜ $\bar{v}(\phi) =$ T: $\langle \phi \rangle^v = \phi = \neg \psi = \langle \psi \rangle^v$

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INDUCTION STEP 4 cases

$\phi = \neg \psi$

induction hypothesis: $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

➜ $\bar{v}(\phi) =$ T: $\langle \phi \rangle^v = \phi = \neg\psi = \langle \psi \rangle^v$

➜ $\bar{v}(\phi) =$ F: $\langle \phi \rangle^v = \neg\phi = \neg\neg\psi$ and $\langle \psi \rangle^v = \psi$

extend Π with $\neg\neg$i to get proof of $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \phi \rangle^v$

INDUCTION STEP

$\phi = \psi_1 \wedge \psi_2$

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➜ $\bar{v}(\psi_1) = \mathsf{T}$, $\bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \wedge \psi_2 = \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$

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- $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \wedge \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

INDUCTION STEP $\phi = \psi_1 \wedge \psi_2$

let $q_1, \ldots, q_l$ be all atoms in ψ_1 and $r_1, \ldots, r_k$ all atoms in ψ_2

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$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i)

- $\bar{v}(\psi_1) = \mathrm{T}, \bar{v}(\psi_2) = \mathrm{T}$: $\langle \phi \rangle^v = \psi_1 \wedge \psi_2 = \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$
- $\bar{v}(\psi_1) = \mathrm{T}, \bar{v}(\psi_2) = \mathrm{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \wedge \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \wedge \psi_2)$ is valid

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- ➜ $\bar{v}(\psi_1) = $ T, $\bar{v}(\psi_2) = $ T: $\langle \phi \rangle^v = \psi_1 \wedge \psi_2 = \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$
- ➜ $\bar{v}(\psi_1) = $ T, $\bar{v}(\psi_2) = $ F: $\langle \phi \rangle^v = \neg(\psi_1 \wedge \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \wedge \psi_2)$ is valid
- ➜ $\bar{v}(\psi_1) = $ F, $\bar{v}(\psi_2) = $ T: $\langle \phi \rangle^v = \neg(\psi_1 \wedge \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \neg\psi_1 \wedge \psi_2$

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- ➜ $\bar{v}(\psi_1) = \mathrm{T}, \bar{v}(\psi_2) = \mathrm{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \wedge \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

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INDUCTION STEP $\phi = \psi_1 \vee \psi_2$

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  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \psi_1 \vee \psi_2$ is valid

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$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i)

- $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \vee \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \psi_2$
  to prove: $\psi_1 \wedge \psi_2 \vdash \psi_1 \vee \psi_2$ is valid
- $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \psi_1 \vee \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$
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- $\bar{v}(\psi_1) = $ T, $\bar{v}(\psi_2) = $ F: $\langle \phi \rangle^v = \psi_1 \vee \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \psi_1 \vee \psi_2$ is valid

- $\bar{v}(\psi_1) = $ F, $\bar{v}(\psi_2) = $ T: $\langle \phi \rangle^v = \psi_1 \vee \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \neg\psi_1 \wedge \psi_2$

  to prove: $\neg\psi_1 \wedge \psi_2 \vdash \psi_1 \vee \psi_2$ is valid

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  to prove: $\neg\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \vee \psi_2)$ is valid

INDUCTION STEP $\phi = \psi_1 \rightarrow \psi_2$

let $q_1, \ldots, q_l$ be all atoms in ψ_1 and $r_1, \ldots, r_k$ all atoms in ψ_2

IH: $\langle q_1 \rangle^v, \ldots, \langle q_l \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle r_1 \rangle^v, \ldots, \langle r_k \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid

$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

INDUCTION STEP $\phi = \psi_1 \to \psi_2$

let $q_1, \ldots, q_l$ be all atoms in ψ_1 and $r_1, \ldots, r_k$ all atoms in ψ_2

IH: $\langle q_1 \rangle^v, \ldots, \langle q_l \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle r_1 \rangle^v, \ldots, \langle r_k \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid

$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

➜ $\bar{v}(\psi_1) =$ T, $\bar{v}(\psi_2) =$ T: $\langle \phi \rangle^v = \psi_1 \to \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \psi_2$

INDUCTION STEP $\phi = \psi_1 \rightarrow \psi_2$

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- ➜ $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \psi_2$

  to prove: $\psi_1 \wedge \psi_2 \vdash \psi_1 \rightarrow \psi_2$ is valid

- ➜ $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \rightarrow \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

INDUCTION STEP $\phi = \psi_1 \rightarrow \psi_2$

let $q_1, \dots, q_l$ be all atoms in ψ_1 and $r_1, \dots, r_k$ all atoms in ψ_2

IH: $\langle q_1 \rangle^v, \dots, \langle q_l \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle r_1 \rangle^v, \dots, \langle r_k \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid

$\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

- $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \psi_2$

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let $q_1, \ldots, q_l$ be all atoms in ψ_1 and $r_1, \ldots, r_k$ all atoms in ψ_2

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$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

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  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \rightarrow \psi_2)$ is valid

- ➜ $\bar{v}(\psi_1) = \mathsf{F}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \neg\psi_1 \wedge \psi_2$

  to prove: $\neg\psi_1 \wedge \psi_2 \vdash \psi_1 \rightarrow \psi_2$ is valid

INDUCTION STEP $\phi = \psi_1 \rightarrow \psi_2$

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$\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

- ➜ $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \psi_2$

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- ➜ $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \rightarrow \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

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$\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (as before)

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- $\bar{v}(\psi_1) = \mathsf{T}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \neg(\psi_1 \rightarrow \psi_2)$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \psi_1 \wedge \neg\psi_2$

  to prove: $\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \rightarrow \psi_2)$ is valid

- $\bar{v}(\psi_1) = \mathsf{F}, \bar{v}(\psi_2) = \mathsf{T}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \neg\psi_1 \wedge \psi_2$

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- $\bar{v}(\psi_1) = \mathsf{F}, \bar{v}(\psi_2) = \mathsf{F}$: $\langle \phi \rangle^v = \psi_1 \rightarrow \psi_2$, $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v = \neg\psi_1 \wedge \neg\psi_2$

  to prove: $\neg\psi_1 \wedge \neg\psi_2 \vdash \psi_1 \rightarrow \psi_2$ is valid

□

② $\models \phi \quad \Longrightarrow \quad \vdash \phi$

CONSTRUCTION

suppose $\models \phi$

② $\models \phi \implies \vdash \phi$

CONSTRUCTION

suppose $\models \phi$

➜ $\forall$ valuation v $\langle \phi \rangle^v = \phi$

② $\models \phi \quad \Longrightarrow \quad \vdash \phi$

CONSTRUCTION

suppose $\models \phi$

➜ $\forall$ valuation v $\langle\phi\rangle^v = \phi$

➜ $\forall$ valuation v $\langle p_1\rangle^v, \ldots, \langle p_n\rangle^v \vdash \phi$ is valid sequent

② $\models \phi \quad \Longrightarrow \quad \vdash \phi$

CONSTRUCTION

suppose $\models \phi$

➜ $\forall$ valuation v $\quad \langle \phi \rangle^v = \phi$

➜ $\forall$ valuation v $\quad \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \phi$ is valid sequent

➜ combine all proofs of these sequents into proof of

$$\vdash \phi$$

by applying LEM $2^n - 1$ times

$$p \wedge q \to q$$

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \wedge q \rightarrow q$	
v_2	T	F	$p, \neg q \vdash p \wedge q \rightarrow q$	
v_3	F	T	$\neg p, q \vdash p \wedge q \rightarrow q$	
v_4	F	F	$\neg p, \neg q \vdash p \wedge q \rightarrow q$	

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \wedge q \to q$	Π_1
v_2	T	F	$p, \neg q \vdash p \wedge q \to q$	Π_2
v_3	F	T	$\neg p, q \vdash p \wedge q \to q$	Π_3
v_4	F	F	$\neg p, \neg q \vdash p \wedge q \to q$	Π_4

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \wedge q \rightarrow q$	Π_1
v_2	T	F	$p, \neg q \vdash p \wedge q \rightarrow q$	Π_2
v_3	F	T	$\neg p, q \vdash p \wedge q \rightarrow q$	Π_3
v_4	F	F	$\neg p, \neg q \vdash p \wedge q \rightarrow q$	Π_4

$p \vee \neg p$ LEM

- p ass
- $q \vee \neg q$ LEM
  - q ass
  - $\cdots \Pi_1 \cdots$
  - $p \wedge q \rightarrow q$

  - $\neg q$ ass
  - $\cdots \Pi_2 \cdots$
  - $p \wedge q \rightarrow q$
- $p \wedge q \rightarrow q$ $\vee$e

- $\neg p$ ass
- $q \vee \neg q$ LEM
  - q ass
  - $\cdots \Pi_3 \cdots$
  - $p \wedge q \rightarrow q$

  - $\neg q$ ass
  - $\cdots \Pi_4 \cdots$
  - $p \wedge q \rightarrow q$
- $p \wedge q \rightarrow q$ $\vee$e

$p \wedge q \rightarrow q$ $\vee$e

Natural Deduction for Propositional Logic

	introduction	elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$		$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

derived proof rules

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \text{MT}$$

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC}$$

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

Predicate Logic

concept	notation	intended meaning
predicate symbols	P, Q, R, A, B	relations over domain
function symbols	f, g, h, a, b	functions over domain
variables	x, y, z	(unspecified) elements of domain
quantifiers	$\forall, \exists$	for all, for some
connectives	$\neg, \wedge, \vee, \rightarrow$	

REMARKS

- ➜ function and predicate symbols take a fixed number of arguments (arity)
- ➜ function and predicate symbols of arity 0 are called constants
- ➜ $=$ is designated predicate symbol of arity 2

Natural Deduction Rules for Equality

➩ equality introduction

$$\frac{}{t = t} \; =\mathrm{i}$$

Natural Deduction Rules for Equality

- equality introduction

$$\frac{}{t = t} =\text{i}$$

- equality elimination (replace equals by equals)

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]} =\text{e}$$

provided t_1 and t_2 are free for x in ϕ

$s = t \vdash t = s$ is valid:

1	$s = t$	premise
2	$s = s$	$=$i
3	$t = s$	$=$e 1, 2

$s = t \vdash t = s$ is valid:

1	$s = t$	premise		
2	$s = s$	$=$i		
3	$t = s$	$=$e 1, 2		with $\phi = (x = s)$, $t_1 = s$, $t_2 = t$

$s = t \vdash t = s$ is valid:

1	$s = t$	premise	
2	$s = s$	=i	
3	$t = s$	=e 1, 2	with $\phi = (x = s)$, $t_1 = s$, $t_2 = t$

$s = t, t = u \vdash s = u$ is valid:

1	$s = t$	premise	
2	$t = u$	premise	
3	$s = u$	=e 2, 1	with $\phi = (s = x)$, $t_1 = t$, $t_2 = u$

Natural Deduction Rules for Universal Quantification

- $\forall$ elimination

$$\frac{\forall x\ \phi}{\phi[t/x]}\ \forall\text{e}$$

provided t is free for x in ϕ

Natural Deduction Rules for Universal Quantification

- $\forall$ elimination

$$\frac{\forall x\ \phi}{\phi[t/x]}\ \forall\mathrm{e}$$

provided t is free for x in ϕ

- $\forall$ introduction

$$\frac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad \phi[x_0/x] \end{array}}}{\forall x\ \phi}\ \forall\mathrm{i}$$

where x_0 is fresh variable that is used only inside box

$\forall x\ (P(x) \to Q(x)), \forall x\ P(x) \vdash \forall x\ Q(x)$ is valid:

1		$\forall x\ (P(x) \to Q(x))$	premise
2		$\forall x\ P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	∀e 1
4		$P(x_0)$	∀e 2
5		$Q(x_0)$	→e 3, 4
7		$\forall x\ Q(x)$	∀i 3–5

$P \to \forall x\, Q(x) \vdash \forall x\, (P \to Q(x))$ is valid:

1	$P \to \forall x\, Q(x)$	premise
2	$x0$	
3	P	assumption
4	$\forall x\, Q(x)$	$\to$e 1, 3
5	$Q(x_0)$	$\forall$e 4
6	$P \to Q(x_0)$	$\to$i 3–5
7	$\forall x\, (P \to Q(x))$	$\forall$i 2–6

Natural Deduction Rules for Existential Quantification

➪ ∃ introduction

$$\frac{\phi[t/x]}{\exists x\ \phi}\ \exists\text{i}$$

provided t is free for x in ϕ

Natural Deduction Rules for Existential Quantification

- ∃ introduction

$$\frac{\phi[t/x]}{\exists x\ \phi}\ \exists\text{i}$$

provided t is free for x in ϕ

- ∃ elimination

$$\frac{\exists x\ \phi \qquad \boxed{\begin{array}{ll} x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists\text{e}$$

where x_0 is fresh variable that is used only inside box

$\forall x\ \phi \vdash \exists x\ \phi$ is valid:

1	$\forall x\ \phi$	premise
2	$\phi[x/x]$	$\forall$e 1
3	$\exists x\ \phi$	$\exists$i 2

$\forall x\ \phi \vdash \exists x\ \phi$ is valid:

1	$\forall x\ \phi$	premise
2	$\phi[x/x]$	$\forall$e 1
3	$\exists x\ \phi$	$\exists$i 2

$\forall x\ (P(x) \rightarrow Q(x)), \exists x\ P(x) \vdash \exists x\ Q(x)$ is valid:

1		$\forall x\ (P(x) \rightarrow Q(x))$	premise
2		$\exists x\ P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
5		$Q(x_0)$	$\rightarrow$e 4, 3
6		$\exists x\ Q(x)$	$\exists$i 5
7		$\exists x\ Q(x)$	$\exists$e 2, 3–6

$\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$ is valid:

1		$\exists x\, P(x)$	premise
2		$\forall x\, \forall y\, (P(x) \to Q(y))$	premise
3	y_0		
4	x_0	$P(x_0)$	assumption
5		$\forall y\, (P(x_0) \to Q(y))$	$\forall$e 2
6		$P(x_0) \to Q(y_0)$	$\forall$e 5
7		$Q(y_0)$	$\to$e 6, 4
8		$Q(y_0)$	$\exists$e 1, 4–7
9		$\forall y\, Q(y)$	$\forall$i 3–8

$\neg\forall x\, \phi \vdash \exists x\, \neg\phi$ is valid:

1	$\neg\forall x\, \phi$	premise
2	$\neg\exists x\, \neg\phi$	assumption
3	x_0	
4	$\neg\phi[x_0/x]$	assumption
5	$\exists x\, \neg\phi$	$\exists$i 4
6	$\bot$	$\neg$e 5, 2
7	$\phi[x_0/x]$	PBC 4–6
8	$\forall x\, \phi$	$\forall$i 3–7
9	$\bot$	$\neg$e 8, 1
10	$\exists x\, \neg\phi$	PBC 2–9

Quantifier Equivalencies

THEOREM

$$\neg\forall x\,\phi \quad \dashv\vdash \quad \exists x\,\neg\phi \qquad\qquad \neg\exists x\,\phi \quad \dashv\vdash \quad \forall x\,\neg\phi$$

$$\forall x\,\phi \wedge \forall x\,\psi \quad \dashv\vdash \quad \forall x\,(\phi \wedge \psi) \qquad\qquad \exists x\,\phi \vee \exists x\,\psi \quad \dashv\vdash \quad \exists x\,(\phi \vee \psi)$$

$$\forall x\,\forall y\,\phi \quad \dashv\vdash \quad \forall y\,\forall x\,\phi \qquad\qquad \exists x\,\exists y\,\phi \quad \dashv\vdash \quad \exists y\,\exists x\,\phi$$

if x is not free in ψ then

$$\forall x\,\phi \wedge \psi \quad \dashv\vdash \quad \forall x\,(\phi \wedge \psi) \qquad\qquad \forall x\,\phi \vee \psi \quad \dashv\vdash \quad \forall x\,(\phi \vee \psi)$$

$$\exists x\,\phi \wedge \psi \quad \dashv\vdash \quad \exists x\,(\phi \wedge \psi) \qquad\qquad \exists x\,\phi \vee \psi \quad \dashv\vdash \quad \exists x\,(\phi \vee \psi)$$

$$\forall x\,(\psi \rightarrow \phi) \quad \dashv\vdash \quad \psi \rightarrow \forall x\,\phi \qquad\qquad \forall x\,(\phi \rightarrow \psi) \quad \dashv\vdash \quad \exists x\,\phi \rightarrow \psi$$

$$\exists x\,(\psi \rightarrow \phi) \quad \dashv\vdash \quad \psi \rightarrow \exists x\,\phi \qquad\qquad \exists x\,(\phi \rightarrow \psi) \quad \dashv\vdash \quad \forall x\,\phi \rightarrow \psi$$

$\exists x\ \neg\phi \vdash \neg\forall x\ \phi$ is valid:

1		$\exists x\ \neg\phi$	premise
2		$\forall x\ \phi$	assumption
3	x_0	$(\neg\phi)[x_0/x]$	assumption
4		$\neg(\phi[x_0/x])$	identical
5		$\phi[x_0/x]$	$\forall$e 2
6		$\bot$	$\neg$e 5, 4
7		$\bot$	$\exists$e 1, 3–6
8		$\neg\forall x\ \phi$	$\neg$i 2–7

$\exists x\ \phi \lor \exists x\ \psi \vdash \exists x\ (\phi \lor \psi)$ is valid:

1		$\exists x\ \phi \lor \exists x\ \psi$	premise
2		$\exists x\ \phi$	assumption
3	x_0	$\phi[x_0/x]$	assumption
4		$\phi[x_0/x] \lor \psi[x_0/x]$	$\lor$i$_1$ 3
5		$(\phi \lor \psi)[x_0/x]$	identical
6		$\exists x\ (\phi \lor \psi)$	$\exists$i 5
7		$\exists x\ (\phi \lor \psi)$	$\exists$e 2, 3–6
8		$\exists x\ \psi$	assumption
9	x_0	$\psi[x_0/x]$	assumption
10		$\phi[x_0/x] \lor \psi[x_0/x]$	$\lor$i$_2$ 9
11		$(\phi \lor \psi)[x_0/x]$	identical
12		$\exists x\ (\phi \lor \psi)$	$\exists$i 11
13		$\exists x\ (\phi \lor \psi)$	$\exists$e 8, 9–12
14		$\exists x\ (\phi \lor \psi)$	$\lor$e 1, 2–7, 8–13

$\exists x\,(\phi \vee \psi) \vdash \exists x\,\phi \vee \exists x\,\psi$ is valid:

1		$\exists x\,(\phi \vee \psi)$	premise
2	x_0	$(\phi \vee \psi)[x_0/x]$	assumption
3		$\phi[x_0/x] \vee \psi[x_0/x]$	identical
4		$\phi[x_0/x]$	assumption
5		$\exists x\,\phi$	$\exists$i 4
6		$\exists x\,\phi \vee \exists x\,\psi$	$\vee \mathrm{i}_1$ 5
7		$\psi[x_0/x]$	assumption
8		$\exists x\,\psi$	$\exists$i 7
9		$\exists x\,\phi \vee \exists x\,\psi$	$\vee \mathrm{i}_2$ 8
10		$\exists x\,\phi \vee \exists x\,\psi$	$\vee$e 3, 4–6, 7–9
11		$\exists x\,\phi \vee \exists x\,\psi$	$\exists$e 1, 2–10

$\forall x\, \forall y\, \phi \vdash \forall y\, \forall x\, \phi$ is valid:

1		$\forall x\, \forall y\, \phi$	premise
2	y_0		
3	x_0	$(\forall y\, \phi)[x_0/x]$	$\forall$e 1
4		$\forall y\, (\phi[x_0/x])$	identical
5		$\phi[x_0/x][y_0/y]$	$\forall$e 4
6		$\phi[y_0/y][x_0/x]$	identical
7		$\forall x\, (\phi[y_0/y])$	$\forall$i 3–6
8		$(\forall x\, \phi)[y_0/y]$	identical
9		$\forall y\, \forall x\, \phi$	$\forall$i 2–8

$\exists x \, \exists y \, \phi \vdash \exists y \, \exists x \, \phi$ is valid:

1		$\exists x \, \exists y \, \phi$	premise
2	x_0	$(\exists y \, \phi)[x_0/x]$	assumption
3		$\exists y \, (\phi[x_0/x])$	identical
4	y_0	$\phi[x_0/x][y_0/y]$	assumption
5		$\phi[y_0/y][x_0/x]$	identical
6		$\exists x \, (\phi[y_0/y])$	∃i 5
7		$(\exists x \, \phi)[y_0/y]$	identical
8		$\exists y \, \exists x \, \phi$	∃i 7
9		$\exists y \, \exists x \, \phi$	∃e 3, 4–8
10		$\exists y \, \exists x \, \phi$	∃e 1, 2–9

$\forall x\, \phi \wedge \psi \vdash \forall x\, (\phi \wedge \psi)$ is valid (provided x is not free in ψ):

1		$\forall x\, \phi \wedge \psi$	premise
2		$\forall x\, \phi$	$\wedge e_1$ 1
3		ψ	$\wedge e_2$ 1
4	x_0	$\phi[x_0/x]$	$\forall e$ 2
5		$\phi[x_0/x] \wedge \psi$	$\wedge i$ 4, 3
6		$(\phi \wedge \psi)[x_0/x]$	identical
7		$\forall x\, (\phi \wedge \psi)$	$\forall i$ 4–6

$\forall x\,(\phi \wedge \psi) \vdash \forall x\,\phi \wedge \psi$ is valid (provided x is not free in ψ):

1		$\forall x\,(\phi \wedge \psi)$	premise
2	x_0	$(\phi \wedge \psi)[x_0/x]$	$\forall$e 1
3		$\phi[x_0/x] \wedge \psi$	identical
4		ψ	$\wedge e_2$ 3
5		$\phi[x_0/x]$	$\wedge e_1$ 3
6		$\forall x\,\phi$	$\forall$i 2–5
7		$\forall x\,\phi \wedge \psi$	$\wedge$i 6, 4

$\forall x\, \phi \lor \psi \vdash \forall x\, (\phi \lor \psi)$ is valid (provided x is not free in ψ):

1		$\forall x\, \phi \lor \psi$	premise
2		$\forall x\, \phi$	assumption
3	x_0	$\phi[x_0/x]$	$\forall$e 2
4		$\phi[x_0/x] \lor \psi$	$\lor$i$_1$ 3
5		$(\phi \lor \psi)[x_0/x]$	identical
6		$\forall x\, (\phi \lor \psi)$	$\forall$i 3–5
7		ψ	assumption
8	x_0	$\phi[x_0/x] \lor \psi$	$\lor$i$_2$ 7
9		$(\phi \lor \psi)[x_0/x]$	identical
10		$\forall x\, (\phi \lor \psi)$	$\forall$i 8–9
11		$\forall x\, (\phi \lor \psi)$	$\lor$e 1, 2–6, 7–10

$\forall x\, (\phi \vee \psi) \vdash \forall x\, \phi \vee \psi$ is valid (provided x is not free in ψ):

1		$\forall x\, (\phi \vee \psi)$	premise
2		$\psi \vee \neg\psi$	LEM
3		ψ	assumption
4		$\forall x\, \phi \vee \psi$	$\vee$i$_2$ 3
5		$\neg\psi$	assumption
6	x_0	$(\phi \vee \psi)[x_0/x]$	$\forall$e 1
7		$\phi[x_0/x] \vee \psi$	identical
8		$\phi[x_0/x]$	assumption
9		ψ	assumption
10		$\bot$	$\neg$e 9, 5
11		$\phi[x_0/x]$	$\bot$e 10
12		$\phi[x_0/x]$	$\vee$e 7, 8, 9-11
13		$\forall x\, \phi$	$\forall$i 6–12
14		$\forall x\, \phi \vee \psi$	$\vee$i$_1$ 13
15		$\forall x\, \phi \vee \psi$	$\vee$e 2, 3–4, 5–14

$\forall x\, (\psi \to \phi) \vdash \psi \to \forall x\, \phi$ is valid (provided x is not free in ψ):

1		$\forall x\, (\psi \to \phi)$	premise
2		ψ	assumption
3	x_0	$(\psi \to \phi)[x_0/x]$	$\forall$e 1
4		$\psi \to \phi[x_0/x]$	identical
5		$\phi[x_0/x]$	$\to$e 4, 2
6		$\forall x\, \phi$	$\forall$i 3–5
7		$\psi \to \forall x\, \phi$	$\to$i 2–6

$\psi \to \forall x\, \phi \vdash \forall x\, (\psi \to \phi)$ is valid (provided x is not free in ψ):

1	$\psi \to \forall x\, \phi$	premise
2	x_0	
3	ψ	assumption
4	$\forall x\, \phi$	$\to$e 1, 3
5	$\phi[x_0/x]$	$\forall$e 4
6	$\psi \to \phi[x_0/x]$	$\to$i 3–5
7	$(\psi \to \phi)[x_0/x]$	identical
8	$\forall x\, (\psi \to \phi)$	$\forall$i 2–7

Exercises from last week…

1. First find a reasonable signature (constants, function symbols, predicate symbols) and then write down the following sentences as first order formulas:

   - All humans are mortal.
   - Socrates is a human.
   - There exists human which is immortal.

   Is it possible to find a model for this set of formulas?

$$\forall x\ (human(x) \rightarrow mortal(x))$$
$$human(socrates)$$

$$\begin{aligned}
&\exists x\ (human(x) \land \neg mortal(x)) \\
\Leftrightarrow &\neg \forall x\ \neg (human(x) \land \neg mortal(x)) \\
\Leftrightarrow &\neg \forall x\ \neg (human(x) \land \neg mortal(x)) \\
\Leftrightarrow &\neg \forall x\ (\neg human(x) \lor \neg\neg mortal(x)) \\
\Leftrightarrow &\neg \forall x\ (\neg human(x) \lor \neg\neg mortal(x)) \\
\Leftrightarrow &\neg \forall x\ (human(x) \rightarrow mortal(x))
\end{aligned}$$

2. Write down the following sentences as First Order formulae:

- Some animals eat meat, others are vegetarian.
- Animals eating animals are not vegetarian. Grass, Vegetables, Fruits are Plants. Vegetarians only eat plants.
- Farmers are defined as the manufacturers which produce food and hold animals.
- Cows are animals.
- Belle is a Cow.
- Belle eats Grass.
- Perro is an a Dog.
- Perro eats Belle.

$\exists x\ \exists y\ (animal(x) \wedge eats(x, y) \wedge meat(y))$

$\exists x\ (animal(x) \wedge vegetarian(x))$

$\forall x\ \forall y\ (animal(x) \wedge eats(x, y) \wedge animal(y) \rightarrow \neg vegetarian(x))$

$plant(grass) \wedge plant(vegetable) \wedge plant(fruit)$

$\forall (vegetarian(x) \wedge eats(x, y) \rightarrow plant(y))$

$\forall x\ (farmer(x) \leftrightarrow manufacturer(x) \wedge \exists y(produces(x, y) \wedge food(y)) \wedge \exists z(holds(x, z) \wedge animal(z)))$

$\forall (animal(x) \leftarrow cow(x))$

$cow(belle)$

$eats(belle, grass)$

$dog(perro)$

$eats(perro, belle)$

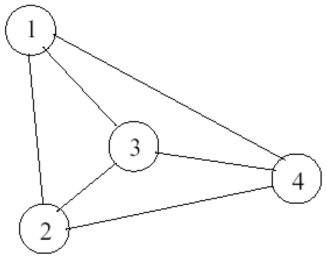
Remark: since the theory doesn't state that perro is an animal, a model where perro is a vegetarian is possible! However, such a model would have strange consequences: $\neg animal(perro), plant(cow)$
This becomes clearer when we rewrite the third and the fifth rule…

3. Write down the following sentences about a graph as First Order formulae. Use the binary predicates $edge/2$, $hasColor/2$ and the unary predicate $node/1$:

- Each node has either the color green, red or blue.
- No two nodes which are connected have the same color.

$$\forall(hasColor(x, green) \vee hasColor(x, red) \vee hasColor(x, blue) \leftarrow node(x))$$

$$\forall(\neg(hasColor(x, z) \wedge hasColor(y, z)) \leftarrow edge(x, y))$$



$$node(1) \wedge node(2) \wedge node(3) \wedge node(4) \wedge$$
$$edge(1, 2) \wedge edge(1, 3) \wedge edge(1, 4) \wedge$$
$$edge(2, 3) \wedge edge(2, 4) \wedge edge(3, 4)$$

Remarks:

- *This set of formulae has no model!*
- *If you write down these formulae in clausal form, you'll see that the first formula is not a Horn clause, while the second is.*
- *The graph can be written down in clausal form (it is a set of facts)!*

4. Given the following closed First Order formula:

$$\forall x\ \exists y\ gt(y, s(x)) \wedge \forall (a(x, y, z) \rightarrow a(s(x), y, s(z)) \wedge a(x, null, x))$$

(a) Find a model for this formula, i.e. specify an interpretation $\mathcal{I}$ for the alphabet consisting of the variable symbols x, y, z the predicate symbols $gt/2, a/3$, the constant $null$ and the function symbol $s/1$ which evaluates the formula to $true$.

(b) Use the evaluation function $Val^{\mathcal{I}}$ to evaluate the truth value of

$$\exists x \exists y (a(s(s(s(0))), s(x), y) \rightarrow gt(y, s(s(s(x)))))$$

Recall: An Interpretation $\mathcal{I}$ consists of:

- a domain D over which the variables can range
- for each n-ary function symbol f a mapping $f^{\mathcal{I}}$ from $D^n \rightarrow D$
  (particularly each constant is assigned an element of D)
- for each n-ary *predicate symbol* an n-ary *relation* over the domain D

Domain: Natural numbers

Constants/Function symbols:

$null^{\mathcal{I}} = 0$

$s^{\mathcal{I}}(x^{\mathcal{I}}) = x^{\mathcal{I}}+1$

Predicate symbols:

$gt^{\mathcal{I}}(x^{\mathcal{I}},y^{\mathcal{I}})$ = true iff $x^{\mathcal{I}} > y^{\mathcal{I}}$

$a^{\mathcal{I}}(x^{\mathcal{I}},y^{\mathcal{I}},z^{\mathcal{I}})$ = true iff $x^{\mathcal{I}}+y^{\mathcal{I}}=z^{\mathcal{I}}$

Exercises this time…

- Show how LEM and PBC follow from the other rules, i.e. how LEM and PBC can be "emulated" by the other rules, i.e. you can proof one from the other.

- You should try to do some examples for proofs in natural deduction yourselves.
  - Let's start with the ones from the completeness proof:

prove: $\psi_1 \wedge \psi_2 \vdash \psi_1 \vee \psi_2$ is valid

prove: $\psi_1 \wedge \neg\psi_2 \vdash \psi_1 \vee \psi_2$ is valid

prove: $\neg\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \vee \psi_2)$ is valid

prove: $\psi_1 \wedge \neg\psi_2 \vdash \neg(\psi_1 \rightarrow \psi_2)$ is valid

- More Examples will be published on the Website by next Monday, send solutions to me by Nov. 6th