A Logic for Hybrid Rules *

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Abstract

In the ongoing discussion about rule extensions for Ontology languages on the Semantic Web a recurring issue is how to combine first-order classical logic with nonmonotonic rule languages. Whereas several modular approaches to define a combined semantics for such hybrid knowledge bases focus mainly on decidability issues, we tackle the matter from a more general point of view. In this paper we show how Quantified Equilibrium Logic (QEL) can function as a unified framework that embraces classical logic as well as disjunctive logic programs under the (open) answer set semantics. In the proposed variant of QEL we relax the unique names assumption from earlier versions. Moreover, we show that this framework elegantly captures several modular approaches to nonmonotonic semantics for hybrid knowledge bases.

1 Introduction

In the ongoing discussion about rule extensions for Ontology languages on the Semantic Web a recurring issue is how to combine a first-order classical theory formalising an ontology with a nonmonotonic rule base. Declarative, nonmonotonic rule languages with a well-defined model-theoretic semantics have gained considerable attention and achieved maturity over the last few years due to the sucess of Answer Set Programming (ASP). ASP is a logic programming and knowledge representation paradigm which extends the stable model semantics for logic programs with many useful features such as aggregates, weak constraints and priorities, and is supported by efficient implementations (for an overview see [1]). As a logical foundation for answer set semantics and a tool for logical analysis in ASP, the system of Equilibrium Logic was presented in [12] and further developed in subsequent works (see [13] for an overview and references).

In the quest to provide the formal underpinning for a nonmonotonic rules layer for the Semantic Web which can coexist in a semantically well-defined manner with the Ontology layer, various proposals for combining classical first-order logic with different variants of ASP have been presented in the literature.[1] On the one hand there are approaches which provide an entailment-based query interface in bodies of ASP rules, notably in the form of dl-programs and HEX-programs [5, 4]. On the other hand, there are hybrid approaches which define integrated models on the basis of the single models of a classical theory [16, 17, 18, 9]. In this paper we focus on the latter. Common to these hybrid approaches is the definition of a modular, non-monotonic semantics, based on classical first-order models on the one hand, and answer set semantics on the other. Additionally they require several syntactical restrictions on the use of predicates from the classical part within rules. With further restrictions of the classical part to decidable Description Logics (DLs), these semantics support straightforward implementations using existing DL reasoners as well as ASP engines in a modular fashion.

Considerations on decidability formed the driving force behind the design of these semantics and little effort has been made so far to compare these approaches from a more general perspective. Two main aspects that we address in this paper have remained open so far:

(1) The above-mentioned approaches to hybrid knowledge bases differ not only in terms of syntactic restrictions, but also on the way they deal with more fundamental issues [3] that arise when classical logics meet ASP such as the domain closure and the unique names assumptions. In particular, the current proposals suggest circumventing these issues by either restricting the allowed models of the classical theory, or by using variants of the traditional answer set semantics which cater for open domains and non-

*This research has been partially supported by the Spanish MEC under the project TIC-2003-9001 and the Acción Integrada "Formal Techniques for Reasoning about Ontologies in E-Science", and by the European Commission under the projects Knowledge Web (IST-2004-507482) and DIP (FP6-507483).

[1] Most of these approaches focus on the Description Logics fragments of first-order logic underlying the Web Ontology Language OWL.

unique names. One of the aims of this paper is to clarify the overall relations of these semantic and syntactic variants in hybrid knowledge bases.

(2) Nonmonotonic models for hybrid knowledge bases proposed so far are defined in a modular fashion which has advantages in terms of defining new algorithms based on existing algorithms for DL and ASP satisfiabiliy. However, a single underlying logic for hybrid knowledge bases which would allow one to capture notions of equivalence between combined knowledge bases in a standard way is still missing.

Quantified Equilibrium Logic (QEL) can serve to tackle both these issues in an elegant manner. First, QEL captures the semantics of generalised open answer sets [9], the most general semantic variant of ASP to date. Second, QEL allows for a direct embedding of classical theories and thus provides a uniform logical foundation for hybrid knowledge bases: As it turns out, the equilibrium models of the combined knowledge base coincide exactly with the modular nonmonotonic models.

The remainder of this paper makes the following contributions: Section 2 recalls some basics of classical first-order logic; Section 3 reformulates different variants of answer set semantics introduced in the literature using a common notation. Moreover, we discuss some correspondences and discrepancies between these variants. Next, we generalise the definitions of hybrid knowledge bases and nonmonotonic semantics from the literature in Section 4 where we also compare existing approaches. We introduce QEL in Section 5 and clarify its relation to ASP. Section 6 describes an embedding of hybrid knowledge bases into QEL and establishes the main result that equilibrium models correspond to nonmonotonic models of hybrid KBs. As a consequence we can capture in QEL the main semantical approaches to hybrid KBs discussed in the paper. We discuss some implications of these results and further work in Section 7 and conclude the paper in Section 8.

2 First-Order Logic (FOL)

A *function-free first-order language* $\mathcal{L} = \langle C, P \rangle$ with equality consists of a sets of constants C and predicate symbols P. Moreover, we assume a fixed countably infinite set of variables, the symbols, '$\rightarrow$', '$\vee$', '$\wedge$', '$\neg$', '$\exists$', '$\forall$', and auxiliary parentheses '(','$)$'. Each predicate symbol $p \in P$ has an assigned arity $ar(p)$. Atoms and formulas are constructed as usual; if D is a non-empty set, we denote by $At_D(C, P)$ the set of atomic sentences of $\mathcal{L} = \langle C, P \rangle$ with additional constant symbols for each element of D. Closed formulas, or *sentences*, are those where each variable is bound by some quantifier. A *theory* $\mathcal{T}$ is a set of sentences. Variable-free atoms, formulas, or theories are also called *ground*.

Given a first-order language $\mathcal{L}$, an $\mathcal{L}$-structure consists of a pair $\mathcal{I} = \langle U, I \rangle$, where the *universe* $U = (D, \sigma)$ consists of a non-empty domain D and a function which assigns a domain value to each constant, $\sigma \colon C \cup D \rightarrow D$ such that $\sigma(d) = d$ for all $d \in D$. We call $d \in D$ an *unnamed* individual if there is no $c \in C$ such that $\sigma(c) = d$. The function I assigns a relation $p^I \subseteq D^n$ to each n-ary predicate symbol $p \in P$ and is called the *$\mathcal{L}$-interpretation over D* . The designated binary predicate symbol eq, occasionally written '$=$' in infix notation, is assumed to be associated the fixed interpretation function $eq^I = \{(d, d) \mid d \in D\}$.

An $\mathcal{L}$-structure $\mathcal{I} = \langle U, I \rangle$ *satisfies* an atom $p(d_1, \ldots, d_n)$ of $At_D(C, P)$, written $\mathcal{I} \models p(d_1, \ldots, d_n)$, iff $(\sigma(d_1), \ldots, \sigma(d_n)) \in p^I$. This is extended as usual to sentences and theories.

An $\mathcal{L}$-structure $\mathcal{I}$ is called a *model* of an atom (sentence, theory, respectively) φ, written $\mathcal{I} \models \varphi$, if it satisfies φ. We say that a theory $\mathcal{T}$ *entails* a sentence φ, written $\mathcal{T} \models \varphi$, if every model of $\mathcal{T}$ is also a model of φ. A theory is *consistent* if it has a model.

In the context of logic programs, often the following assumptions with respect to structures are made:

1. In case σ is surjective, that is, there are no unnamed individuals in D, we say that the *parameter names assumption (PNA)* applies.
2. In case σ is injective we say that the *unique names assumption (UNA)* applies.

The combination of PNA and UNA is called the *standard names assumption* (SNA), that is, σ is a bijection. Consequently, we will speak about PNA-, UNA-, or SNA-structures, (or PNA-, UNA-, or SNA-models, respectively) depending on σ.

An $\mathcal{L}$-interpretation I over D can be defined as a subset of $At_D(C, P)$. So, we can define a subset relation for $\mathcal{L}$-structures $\mathcal{I}_1 = \langle (D, \sigma_1), I_1 \rangle$ and $\mathcal{I}_2 = \langle (D, \sigma_2), I_2 \rangle$ over the same domain by setting $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if $I_1 \subseteq I_2$.[2] Whenever we speak about subset minimality of models/structures in the following, we thus mean minimality among all models/structures over the same domain.

3 Answer Set Semantics

In this paper we assume non-ground disjunctive logic programs with negation allowed in rule heads and bodies under the answer set semantics [10].[3] Syntactically, a program $\mathcal{P}$ consists of a set of rules of the form

$$a_1 \vee a_2 \vee \ldots \vee a_k \vee \neg a_{k+1} \vee \ldots \vee \neg a_l \leftarrow b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n \qquad (1)$$

[2] Note that this is not the substructure or submodel relation in classical model theory, which holds between a structure and its restriction to a subdomain.

[3] By $\neg$ we mean negation as failure and not classical, or strong negation which is also sometimes considered in ASP.

where a_i ($i \in \{1,\dots,l\}$) and b_j ($j \in \{1,\dots,n\}$) are atoms, called head (body, respectively) atoms of the rule, in a function-free first-order language $\mathcal{L} = \langle C, P\rangle$ without equality. By $C_\mathcal{P} \subseteq C$ we denote the constants appearing in $\mathcal{P}$.

A rule with $k = l$ and $m = n$ is called *positive*. Rules where each variable appears in at least one positive body atom are called *safe*. A program is *positive* (*safe*) if all its rules are positive (safe).

For the purposes of this paper, we give a slightly generalised definition of the common notion of the *grounding* of a program:

Definition 1. *The* grounding $gr_U(\mathcal{P})$ *of* $\mathcal{P}$ *wrt. to a universe* $U = (D, \sigma)$ *– sometimes called pre-interpretation – denotes the set of all rules obtained as follows: For* $r \in \mathcal{P}$*, replace (i) each constant* c *appearing in* r *with* $\sigma(c)$ *and (ii) each variable with some element in* D*.*

Observe that thus $gr_U(\mathcal{P})$ is a ground program over the atoms in $At_D(C, P)$.

For a ground program $\mathcal{P}$ and first-order structure $\mathcal{I}$, the *reduct* $\mathcal{P}^\mathcal{I}$ consists of all rules

$$a_1 \vee a_2 \vee \ldots \vee a_k \leftarrow b_1, \ldots, b_m$$

obtained from rules of the form (1) in $\mathcal{P}$ such that $\mathcal{I} \models a_i$ for all $k < i \leq l$ and $\mathcal{I} \not\models b_j$ for all $m < j \leq n$.

Answer set semantics is usually defined in terms of *Herbrand structures* over $\mathcal{L} = \langle C, P\rangle$. Herbrand structures have a fixed universe, the so-called *Herbrand universe* $\mathcal{H} = (C, id)$, where id is the identity function. For an Herbrand structure $\mathcal{I} = \langle \mathcal{H}, I\rangle$, I can be viewed as a subset of the *Herbrand base*, $\mathcal{B}$, which consists of the ground atoms of $\mathcal{L}$. Note that by definition of $\mathcal{H}$, Herbrand structures are SNA-structures.

Definition 2. *An Herbrand structure* $\mathcal{I}$ *is called an* answer set *of* $\mathcal{P}$ *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_\mathcal{H}(\mathcal{P})^\mathcal{I}$.

A variation of this semantics, the open answer set semantics, considers open domains [8], thereby relaxing the PNA: An *extended Herbrand structure* is a first-order structure based on a universe $U = (D, id)$, where $D \supseteq C$. Note that since the assignment is the identity function, the UNA applies.

Definition 3. *An extended Herbrand structure* $\mathcal{I} = \langle U, I\rangle$ *is called an* open answer set *of* $\mathcal{P}$ *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_U(\mathcal{P})^\mathcal{I}$*.*

Open answer set programs allow the use of the equality predicate '=' in the body of rules. However, since this definition of open answer sets adheres to the UNA, one could argue that equality in open answer set programming is purely syntactical. Positive equality predicates in rule bodies can thus be eliminated by simple preprocessing, applying unification. This is not the case for negative occurrences of equality, but, since the interpretation of equality is fixed, these can be eliminated during grounding.

The next variant of answer set semantics we discuss here, introduced by Heymans et al. [9], relaxes the UNA, i.e. arbitrary first-order $\mathcal{L}$-structures are allowed:

Definition 4. *A first-order* $\mathcal{L}$*-structure* $\mathcal{I} = \langle U, I\rangle$ *is called a* generalised open answer set *of* $\mathcal{P}$ *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_U(\mathcal{P})^\mathcal{I}$*.*

An alternative approach to relax the UNA for answer set programming with arbitrary first-order $\mathcal{L}$-structures $\mathcal{I} = \langle U, I\rangle$ has been presented by Rosati in [17]: Instead of grounding into the universe, Rosati grounds with respect to the Herbrand universe $\mathcal{H} = (C, id)$ but redefines minimality of the models of $gr_\mathcal{H}(\mathcal{P})^\mathcal{I}$ wrt. U: Let $\mathcal{I}\!\upharpoonright_\mathcal{H} = \{p(\sigma(c_1), \ldots, \sigma(c_n)) \mid p(c_1, \ldots, c_n) \in \mathcal{B}, \mathcal{I} \models p(c_1, \ldots, c_n)\}$, i.e., the set of atoms from $At_D(C, P)$ true in $\mathcal{I}$ corresponding to ground atoms of $\mathcal{B}$. Given $\mathcal{L}$-structures $\mathcal{I}_1 = (U_1, I_1)$ and $\mathcal{I}_2 = (U_2, I_2)$[4], the relation $\mathcal{I}_1 \subseteq_\mathcal{H} \mathcal{I}_2$ holds if $\mathcal{I}_1\!\upharpoonright_\mathcal{H} \subseteq \mathcal{I}_2\!\upharpoonright_\mathcal{H}$.

Using our notion, we can generalise Rosati's definition of answer sets as follows:

Definition 5. *An* $\mathcal{L}$*-structure* $\mathcal{I}$ *is called a* generalised answer set *of* $\mathcal{P}$ *if* $\mathcal{I}$ *is* $\subseteq_\mathcal{H}$*-minimal among the structures satisfying all rules in* $gr_\mathcal{H}(\mathcal{P})^\mathcal{I}$*.*

The following Lemma (implicit in [7]) states that all atoms from $At_D(C, P)$ satisfied in (generalised open) answer sets of a safe program depend on ground atoms over $C_\mathcal{P}$ only:

Lemma 1. *Let* $\mathcal{P}$ *be a safe program over* $\mathcal{L} = \langle C, P\rangle$ *with* $\mathcal{M} = \langle U, I\rangle$ *a (generalised open) answer set over universe* $U = (D, \sigma)$*. Then, for any atom from* $At_D(C, P)$ *such that* $\mathcal{M} \models p(d_1, \ldots, d_n)$*, there exist* $c_i \in C_\mathcal{P}$ *such that* $\sigma(c_i) = d_i$ *for each* $1 \leq i \leq n$*.*

Proof. First, we observe that any atom $\mathcal{M} \models p(d_1, \ldots, d_n)$ must be derivable from a sequence of rules $(r_0; \ldots; r_l)$ in $gr_U(\mathcal{P})^\mathcal{M}$. We prove the lemma by induction over the length l of this sequence. $l = 0$: Assume $\mathcal{M} \models p(d_1, \ldots, d_n)$, then r_0 must be (by safety) a ground fact in $\mathcal{P}$ such that $p(\sigma(c_1), \ldots, \sigma(c_n)) = p(d_1, \ldots, d_n)$ and $c_1, \ldots, c_n \in C_\mathcal{P}$. As for the induction step, let $p(d_1, \ldots, d_n)$ be inferred by application of rule $r_l \in gr_U(\mathcal{P})^\mathcal{M}$. By safety, again each d_j either stems from a constant $c_j \in C_\mathcal{P}$ such that $\sigma(c_j) = d_j$ which appears in some true head atom of r_l or d_j also appears in

[4] not necessarily over the same domain

a positive body atom $q(\ldots, d_j, \ldots)$ of r_l such that $\mathcal{M} \models q(\ldots, d_j, \ldots)$, derivable by $(r_0; \ldots; r_{l-1})$, which, by the induction hypothesis, proves the existence of a $c_j \in C_{\mathcal{P}}$ with $\sigma(c_j) = d_j$. □

By use of this Lemma, and some other observations we can show the following correspondences between the variants of answer sets just discussed. First, as a straightforward consequence from the definitions we can observe:

Proposition 2. *If $\mathcal{M}$ is an answer set of $\mathcal{P}$ then $\mathcal{M}$ is also an open answer set of $\mathcal{P}$.*

The contrary does not hold in general, as shown by the follwing example:

Example 1. *Let $\mathcal{P} = \{p(a); \quad ok \leftarrow \neg p(x); \quad \leftarrow \neg ok\}$ over $\mathcal{L} = \langle \{a\}, \{p, ok\} \rangle$. We leave it as an exercise to the reader to show that $\mathcal{P}$ is inconsistent under the answer set semantics but $\mathcal{M} = \langle (\{a, c_1\}, id), \{p(a), ok\} \rangle$ is an open answer set.*

Note that the answer sets and open answer sets of *safe* programs coincide as a direct consequence of Lemma 1:

Proposition 3. *$\mathcal{M}$ is an answer set of a safe program $\mathcal{P}$ if and only if $\mathcal{M}$ is an open answer set of $\mathcal{P}$.*

Similarly, on unsafe programs, generalised answer sets and generalised open answer sets do not nesessarily coincide as already shown by Example 1. However, by Lemma 1 we can show the following:

Proposition 4. *Given a safe program $\mathcal{P}$, $\mathcal{M}$ is a generalised open answer set of $\mathcal{P}$ if and only if M is a generalised answer set of $\mathcal{P}$.*

Proof. $\Rightarrow$: Assume $\mathcal{M}$ is a generalised open answer set of $\mathcal{P}$. By Lemma 1 we know that rules in $gr_U(\mathcal{P})^{\mathcal{M}}$ involving unnamed individuals do not contribute to answer sets, since their body is always false. It follows that $\mathcal{M} = \mathcal{M}|_{\mathcal{H}}$ which in turn is a $\subseteq_{\mathcal{H}}$-minimal model of $gr_{\mathcal{H}}(\mathcal{P})^{\mathcal{M}}$. This follows from the observation that each rule in $gr_{\mathcal{H}}(\mathcal{P})^{\mathcal{M}}$ and its corresponding rules in $gr_U(\mathcal{P})^{\mathcal{M}}$ are satisfied under the same models. $\Leftarrow$: Analogously. □

By similar arguments, generalised answer sets and generalised open answer sets coincide in case the parameter name assumption applies:

Proposition 5. *Let $\mathcal{M}$ be a PNA-structure. Then $\mathcal{M}$ is a generalised answer set of $\mathcal{P}$ if and only if $\mathcal{M}$ is a generalised open answer of $\mathcal{P}$.*

A final straightforward consequence from the previously shown results is that under SNA, consistency with respect to all semantics introduced so far boils down to consistency under the original definition of answer sets:

Proposition 6. *A program $\mathcal{P}$ has an answer set if and only if $\mathcal{P}$ has a generalised open answer under the SNA.*

We note that the actual answer sets under SNA differ from the original answer sets since also non-Herbrand structures are allowed.

Finally, we observe that there are programs which have generalised (open) answer sets but do not have (open) answer sets, even on safe programs, as shown by the following simple example:

Example 2. *Let $\mathcal{P} = \{p(a); \quad \leftarrow \neg p(b)\}$ over $\mathcal{L} = \langle \{a, b\}, \{p\} \rangle$. This program is ground, thus obviously safe. However, although $\mathcal{P}$ has a generalised (open) answer set – the reader may verify this for instance considering the one-element universe $U = (\{d\}, \sigma)$, where $\sigma(a) = \sigma(b) = d$ – it is inconsistent under the open answer set semantics.*

4 Hybrid Knowledge Bases

We now turn to the concept of hybrid knowledge base which combines classical theories with the various notions of answer sets from the previous section. Based on several definitions in the literature [16, 17, 18, 9], we extract a generalised notion of hybrid knowledge base. We then compare and discuss the differences between the various definitions which mainly affect decidability issues but do not change the general semantics. This will allow us to base our embedding into Quantified Equilibrium Logic on a unified definition in the subsequent sections.

A *hybrid knowledge base* $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ consists of a classical first-order theory $\mathcal{T}$ (also called the *structural* part of $\mathcal{K}$) over the function-free language $\mathcal{L}_{\mathcal{T}} = \langle C, P_{\mathcal{T}} \rangle$ and a program $\mathcal{P}$ (also called *rules* part of $\mathcal{K}$) over the language $\mathcal{L} = \langle C, P_{\mathcal{T}} \cup P_{\mathcal{P}} \rangle$, where $P_{\mathcal{T}} \cap P_{\mathcal{P}} = \emptyset$, i.e.

- $\mathcal{T}$ and $\mathcal{P}$ talk about the same constants, and
- the predicate symbols in $\mathcal{P}$ are a superset of the predicate symbols in $\mathcal{L}_{\mathcal{T}}$.

By $\mathcal{L}_{\mathcal{P}} = \langle C, P_{\mathcal{P}} \rangle$ we denote the restricted language of $\mathcal{P}$ without the predicate symbols in $P_{\mathcal{T}}$.

Hybrid knowledge bases provide a unified semantics for a combination of classical theories and nonmonotonic rules. Intuitively, $\mathcal{L}_{\mathcal{T}}$ and $\mathcal{L}_{\mathcal{P}}$ are interpreted separately, where the predicates in $\mathcal{L}_{\mathcal{T}}$ are interpreted classically, whereas the predicates in $\mathcal{L}_{\mathcal{P}}$ are interpreted nonmonotonically under the (generalised open) answer set semantics.

The semantics we present in the following generalises the nonmonotonic semantics defined in [16, 17, 18, 9]. We will discuss particular features and restrictions of each of these approaches separately later on.

We do not consider the alternative classical semantics defined in [16, 17, 18], as these are straightforward.

If $\mathcal{P}$ is a ground program and $\mathcal{I} = \langle U, I \rangle$ is an $\mathcal{L}$-structure, the *projection* of $\mathcal{P}$ wrt $\mathcal{I}$ is denoted by $\Pi(\mathcal{P}, \mathcal{I})$ and defined as follows: for each rule $r \in \mathcal{P}$ define r^{Π} by

1. $r^{\Pi} = \emptyset$ if there is a literal over $At_D(C, P_T)$ in the head of r of form $p(t)$[5] such that $p(\sigma(t)) \in I$ or of form $\neg p(t)$ with $p(\sigma(t)) \notin I$;
2. $r^{\Pi} = \emptyset$ if there is a literal over $At_D(C, P_T)$ in the body of r of form $p(t)$ such that $p(\sigma(t)) \notin I$ or of form $\neg p(t)$ such that $p(\sigma(t)) \in I$;
3. otherwise r^{Π} results from r by deleting all occurrences of literals from $\mathcal{L}_{\mathcal{T}}$.

Then we set $\Pi(\mathcal{P}, \mathcal{I}) = \{r^{\Pi} : r \in \mathcal{P}\}$. Intuitively, the projection "evaluates" all classical literals in $\mathcal{P}$ with respect to $\mathcal{I}$. *Nonmonotonic models* of hybrid knowledge bases are defined with respect to this projection:

If $\mathcal{I}$ is an $\mathcal{L}'$-structure we denote by $\mathcal{I}|_{\mathcal{L}}$ the restriction of $\mathcal{I}$ to a sublanguage $\mathcal{L} \subseteq \mathcal{L}'$.

Definition 6. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be a hybrid knowledge base. An NM-model $\mathcal{M} = \langle U, I \rangle$ (where $U = (D, \sigma)$) of a hybrid knowledge base $\mathcal{K}$ is a first-order $\mathcal{L}$-structure such that $\mathcal{M}|_{\mathcal{L}_{\mathcal{T}}}$ is a model of $\mathcal{T}$ and $\mathcal{M}|_{\mathcal{L}_{\mathcal{P}}}$ is a generalised open answer set of* $\Pi(gr_U(\mathcal{P}), \mathcal{M})$.

Analogously to first-order models, we speak about PNA-, UNA-, or SNA-NM-models, depending on σ.

We will now turn to the question under which restrictions the various definitions of NM-models for hybrid KBs in the literature are subsumed by this general definition.

4.1 r-hybrid KBs

Originally, the definition of hybrid knowledge base was introduced by Rosati [16] under the label *r-hybrid* knowledge base. Syntactically, *r-hybrid* KBs only allow positive atoms in rule heads, i.e. for rules of the form (1) we have $l = k$, and do not allow atoms from $\mathcal{L}_{\mathcal{T}}$ to occur negated.[6] Moreover, Rosati deploys a restriction which is stronger than standard safety: each variable must appear in at least one positive body atom from $\mathcal{L}_{\mathcal{P}}$. Thus, we call this condition *$\mathcal{L}_{\mathcal{P}}$-safe* in the following.

Semantically, Rosati assumes (an infinite supply of) standard names, i.e. C is countably inifinite, and normal answer sets in his version of NM-models:

Definition 7. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be an r-hybrid knowledge base consisting of a theory $\mathcal{T}$ and an $\mathcal{L}_{\mathcal{P}}$-safe program $\mathcal{P}$. An* r-NM-model *$\mathcal{M} = \langle U, I \rangle$ of $\mathcal{K}$ is a first-order $\mathcal{L}$-SNA-structure such that $\mathcal{M}|_{\mathcal{L}_{\mathcal{T}}}$ is a model of $\mathcal{T}$ and $\mathcal{M}|_{\mathcal{L}_{\mathcal{P}}}$ is an answer set of* $\Pi(gr_U(\mathcal{P}), \mathcal{M})$.

In virtue of $\mathcal{L}_{\mathcal{P}}$-safety, we observe that r-NM-model consistency coincides with SNA-NM-model consistency on r-hybrid knowledge bases, by an argument similar to that of Lemma 1 and Proposition 6.

The syntactic restrictions in r-hybrid knowledge bases are mostly motivated as they allow the definition of a sound and complete algorithm presented in [16] for deciding NM-consistency on r-hybrid $\mathcal{K}$ based on standard solvers under the assumption that $\mathcal{T}$ is a theory in a fragment of FOL where satisfiability is decidable.[7] As $\mathcal{L}_{\mathcal{P}}$-safety guarantees that also $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ is safe, by Lemma 1 it is sufficient to ground only over constants in $C_{\mathcal{P}}$, which is typically done by standard answer set solvers.

4.2 r$^+$-hybrid KBs

Rosati relaxes the UNA for what we will call here *r$^+$-hybrid* knowledge bases in [17]. In this variant the $\mathcal{L}_{\mathcal{P}}$-safety restriction is kept but generalised answer sets under arbitrary interpretations are considered:

Definition 8. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be an r$^+$-hybrid knowledge base consisting of a theory $\mathcal{T}$ and an $\mathcal{L}_{\mathcal{P}}$-safe program $\mathcal{P}$. An* r$^+$-NM-model, *$\mathcal{M} = \langle U, I \rangle$ of $\mathcal{K}$ is a first-order $\mathcal{L}$-structure such that $\mathcal{M}|_{\mathcal{L}_{\mathcal{T}}}$ is a model of $\mathcal{T}$ and $\mathcal{M}|_{\mathcal{L}_{\mathcal{P}}}$ is a generalised answer set of* $\Pi(gr_U(\mathcal{P}), \mathcal{M})$.

Whereas Rosati does not make any claims about PNA in this work, by the fact that $\mathcal{L}_{\mathcal{P}}$-safety guarantees safety of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ and by Proposition 4 we can conclude that r$^+$-NM-models coincide with NM-models on r$^+$-hybrid knowledge bases.

Decidability results are the same as in r-hybrid knowledge bases for this relaxed version. In order to deal with the relaxation of UNA, the algorithm from [16] has to be modified by a preprocessing step which eliminates multiple occurrences of the same variable in the non-ground program, by introducing equalities in the rule bodies. Equality is viewed as a predicate in $\mathcal{L}_{\mathcal{T}}$ in this approach.

4.3 r$_w$-hybrid KBs

Recently, Rosati presented his latest version, which we call here *weakly safe hybrid (r$_w$-hybrid)* knowledge bases [18]. Here, SNA is reintroduced, but safety is relaxed to *weak $\mathcal{L}_{\mathcal{P}}$-safety*, which means that $\mathcal{L}_{\mathcal{P}}$-unsafe variable occurrences are allowed in body atoms from $P_{\mathcal{T}}$ as long as they do not occur in any predicate from $P_{\mathcal{P}}$. The

[5] Note: we use t for a tuple $d_1, \ldots, d_n$ here, for ease of legibility.

[6] Note that by projection, negation of predicates from $P_{\mathcal{T}}$ in rules is treated completely classically, whereas negation of predicates from $P_{\mathcal{P}}$ is treated nonmonotonically. This might be considered unintuitive and therefore a reason why Rosati disallows structural predicates to occur negated.

[7] This particularly applies to the DL underlying OWL.

	SNA	variables	disj.	neg. $\mathcal{L}_\mathcal{T}$ atoms
r-hybrid [16]	yes	$\mathcal{L}_\mathcal{P}$-safe	pos.	no
r$^+$-hybrid [17]	no	$\mathcal{L}_\mathcal{P}$-safe	pos.	no
r$_w$-hybrid [18]	yes	weak $\mathcal{L}_\mathcal{P}$-safe	pos.	no
g-hybrid [9]	no	guarded	neg.*	yes

* g-hybrid allows negation in the head but at most one positive head atom

Table 1. Different variants of hybrid KBs

definition of models follows Definition 7. As the relaxed safety restriction still guarantees safety of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ we can conclude that r-NM-model consistency coincides with SNA-NM-model consistency on r$_w$-hybrid knowledge bases.[8]

The main difference with respect to r-hybrid knowledge bases is that the author shows that consistency of r$_w$-hybrid knowledge bases is decidable in case $\mathcal{T}$ is a theory in an FOL fragment where conjunctive query containment is decidable. This is, according to [18], the case for a number of expressive DLs, and in particular conjectured, but not yet proven, for $\mathcal{SHIQ}$, a DL very close to OWL.

4.4 g-hybrid KBs

The last approach we discuss here was introduced by Heymans et al. [9]. Syntactically, [9] makes a different restriction on variables than Rosati, called *guardedness*. Therefore, the approach is called *g-hybrid*: For all rules with a non-empty body, it is required that all variables occur in one designated positive body atom, the so-called rule guard. Additionally, unsafe choice rules of the form

$$p(c_1, \ldots, c_n) \vee \neg p(c_1, \ldots, c_n) \leftarrow$$

are allowed. Moreover, disjunction in rule heads is limited to at most one positive atom, i.e. for rules of the form (1) we have that $k \leq 1$, but an arbitrary number of negated head atoms is allowed. Another significant difference is that, as opposed to the approaches based on r-hybrid KBs, negative structural predicates are allowed in the rules part within g-hybrid knowledge bases (see also Footnote 6).

Apart from these syntactic restrictions, which are again driven by considerations of decidability, Heymans et al.'s definition of NM-models coincides precisely with our Definition 6, i.e. generalised open answer sets are considered.

Table 4.4 summarizes the different versions of hybrid knowledge bases introduced in the literature.

[8]Strictly read, the safety restriction as defined in [18] also allows unsafe rules in $\Pi(gr_U(\mathcal{P}), \mathcal{M})$, which however is not intended (personal communication with the author).

5 Quantified Equilibrium Logic (QEL)

Equilibrium logic for propositional theories and logic programs was presented in [12] as a foundation for answer set semantics and extended to the first-order case in [14] and in slightly more general, modified form in [15]. For a survey of the main properties of equilibrium logic, see [13]. Usually in quantified equilibrium logic we consider a full first-order language allowing function symbols and we include a second, strong negation operator as occurs in several ASP dialects. For the present purpose of drawing comparisons with approaches to hybrid knowledge bases, it will suffice to consider the function-free language with a single negation symbol, '$\neg$'. So, in particular, we shall work with a quantified version of the logic HT of *here-and-there*. In other respects we follow the treatment of [15].

5.1 General Structures for Quantified Here-and-There Logic

As before we consider function-free first order languages $\mathcal{L} = \langle C, P\rangle$ built over a set of *constant* symbols, C, and a set of *predicate* symbols, P. The set of $\mathcal{L}$-formulas, $\mathcal{L}$-sentences and atomic $\mathcal{L}$-sentences are defined in the usual way.

Again, we only work with *sentences*, and, as in Section 2, by an $\mathcal{L}$-interpretation I over a set D we mean a subset I of $At_D(C, P)$. A *here-and-there* $\mathcal{L}$-structure with static domains or QHT^s-$\mathcal{L}$-*structure* is a tuple $\mathcal{M} = \langle (D, \sigma), I_h, I_t\rangle$ where

- D is a non-empty set, called the *domain* of $\mathcal{M}$.
- σ is a mapping: $C \cup D \to D$ called the *assignment* such that $\sigma(d) = d$ for all $d \in D$. If $D = C$ and $\sigma = id$, $\mathcal{M}$ is a *Herbrand structure*.
- I_h, I_t are $\mathcal{L}$-interpretations over D such that $I_h \subseteq I_t$.

We can think of $\mathcal{M}$ as a structure similar to a first-order classical model, but having two parts or components h and t that correspond to two different points or "worlds", 'here' and 'there' in the sense of Kripke semantics for intuitionistic logic [20], where the worlds are ordered by $h \leq t$. At each world $w \in \{h, t\}$ one verifies a set of atoms I_w in the expanded language for the domain D. We call the model static, since, in contrast to say intuitionistic logic, the same domain serves each of the worlds.[9] Since $h \leq t$, whatever is verified at h remains true at t. The satisfaction relation for $\mathcal{M}$ is defined so as to reflect the two different components, so we write $\mathcal{M}, w \models \varphi$ to denote that φ is true in $\mathcal{M}$ with

[9]Alternatively it is quite common to speak of a logic with *constant* domains. However this is ambiguous since it might suggest that the domain is composed only of constants, which is not intended here.

respect to the w component. Evidently we should require that an atomic sentence is true at w just in case it belongs to the w-interpretation. Formally, if $p(t_1, \ldots, t_n) \in \mathrm{At}_D$ then

$$\mathcal{M}, w \models p(t_1, \ldots, t_n) \quad \text{iff} \quad p(\sigma(t_1), \ldots, \sigma(t_n)) \in I_w. \qquad (2)$$

Then $\models$ is extended recursively as follows[10]:

- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$.
- $\mathcal{M}, t \models \varphi \rightarrow \psi$ iff $\mathcal{M}, t \not\models \varphi$ or $\mathcal{M}, t \models \psi$.
- $\mathcal{M}, h \models \varphi \rightarrow \psi$ iff $\mathcal{M}, t \models \varphi \rightarrow \psi$ and $\mathcal{M}, h \not\models \varphi$ or $\mathcal{M}, h \models \psi$.
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, t \not\models \varphi$.
- $\mathcal{M}, t \models \forall x \varphi(x)$ iff $\mathcal{M}, t \models \varphi(d)$ for all $d \in D$.
- $\mathcal{M}, h \models \forall x \varphi(x)$ iff $\mathcal{M}, t \models \forall x \varphi(x)$ and $\mathcal{M}, h \models \varphi(d)$ for all $d \in D$.
- $\mathcal{M}, w \models \exists x \varphi(x)$ iff $\mathcal{M}, w \models \varphi(d)$ for some $d \in D$.

Truth of a sentence in a model is defined as follows: $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \models \varphi$ for each $w \in \{h, t\}$. A sentence φ is valid if it is true in all models, denoted by $\models \varphi$. A sentence φ is a consequence of a set of sentences Γ if every model of Γ is a model of φ, in symbols $\Gamma \models \varphi$. In a model $\mathcal{M}$ we often use the symbols H and T, possibly with subscripts, to denote the interpretations I_h and I_t respectively; so, an $\mathcal{L}$-structure may be written in the form $\langle U, H, T\rangle$, where $U = (D, \sigma)$.

The resulting logic is called *Quantified Here-and-There Logic with static domains* denoted by $QHT^s_{\mathcal{L}}$. In terms of satisfiability and validity this logic is equivalent to the logic introduced before in [14].

The logic QHT^s can be axiomatised as follows.[11] We start with the usual axioms and rules of intuitionistic propositional logic and add the axiom of Hosoi

$$\alpha \vee (\neg\beta \vee (\alpha \rightarrow \beta))$$

which determines 2-element, here-and-there models. This system is extended to first-order logic (see [14, 15]) by adding the following axioms to obtain the usual non-static version of first-order here-and-there logic:

$$\forall x \neg\neg\alpha(x) \rightarrow \neg\neg\forall x \alpha(x)$$
$$\neg\neg\forall x \alpha(x) \rightarrow \exists x(\alpha(x) \rightarrow \forall x \alpha(x))$$

Finally, we can we add the following axiom for static domains, to obtain $\mathbf{QHT}^s$:

$$\neg\neg\exists x \alpha(x) \rightarrow \exists x \neg\neg\alpha(x)$$

[10] The reader may easily check that the following correspond exactly to the usual Kripke semantics for intuitionistic logic given our assumptions about the two worlds h and t and the single domain D, see eg [20]

[11] We restrict attention here to the logic without equality.

Independence from the language. As usual in first order logic, satisfiability and validity are independent from the language. If $\mathcal{M} = \langle (D, \sigma), H, T\rangle$ is an QHT^s-$\mathcal{L}'$-structure and $\mathcal{L} \subset \mathcal{L}'$, we denote by $\mathcal{M}|_{\mathcal{L}}$ the restriction of $\mathcal{M}$ to the sublanguage $\mathcal{L}$:

$$\mathcal{M}|_{\mathcal{L}} = \langle (D, \sigma|_{\mathcal{L}}), H|_{\mathcal{L}}, T|_{\mathcal{L}}\rangle$$

Proposition 7. *Suppose that $\mathcal{L}' \supset \mathcal{L}$, Γ is a theory in $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}'$-structure such $\mathcal{M} \models \Gamma$. Then $\mathcal{M}|_{\mathcal{L}}$ is a model of Γ in $QHT^s_{\mathcal{L}}$.*

Proposition 8. *Suppose that $\mathcal{L}' \supset \mathcal{L}$ and $\varphi \in \mathcal{L}$. Then φ is valid (resp. satisfiable) in $QHT^s_{\mathcal{L}}$ if and only if is valid (resp. satisfiable) in $QHT^s_{\mathcal{L}'}$.*

Analogous to the case of classical models we can define special kinds of QHT^s models. Let $\mathcal{M} = \langle (D, \sigma), H, T\rangle$ be an $\mathcal{L}$-structure that is a model of a universal theory T. Then, we call $\mathcal{M}$ a PNA-, UNA-, or SNA-model if the restriction of σ to constants in $\mathcal{C}$ is surjective, injective or bijective, respectively.

5.2 Equilibrium Models

As in the propositional case, quantified equilibrium logic is based on a suitable notion of minimal model.

Definition 9. *Among QHT^s-$\mathcal{L}$-structures we define the order $\trianglelefteq$ as follows: $\langle (D, \sigma), H, T\rangle \trianglelefteq \langle (D', \sigma'), H', T'\rangle$ if $D = D'$, $\sigma = \sigma'$, $T = T'$ and $H \subseteq H'$. If the subset relation holds strictly, we write '$\triangleleft$'.*

Definition 10. *Let Γ be a set of sentences and $\mathcal{M} = \langle (D, \sigma), H, T\rangle$ a model of Γ.*

1. *$\mathcal{M}$ is said to be* total *if $H = T$.*
2. *$\mathcal{M}$ is said to be an* equilibrium *model of Γ (or short, we say: "$\mathcal{M}$ is in equilibrium") if it is minimal under $\trianglelefteq$ among models of Γ, and it is total.*

Notice that a total QHT^s model of a theory Γ is equivalent to a classical first order model of Γ.

Proposition 9. *Let Γ be a theory in $\mathcal{L}$ and $\mathcal{M}$ an equilibrium model of Γ in $QHT^s_{\mathcal{L}'}$ with $\mathcal{L}' \supset \mathcal{L}$. Then $\mathcal{M}|_{\mathcal{L}}$ is an equilibrium model of Γ in $QHT^s_{\mathcal{L}}$.*

5.3 Relation to Answer Sets

The above version of QEL is described in more detail in [15]. If we assume all models are UNA-models, we obtain the version of QEL found in [14]. There, the relation of QEL to (ordinary) answer sets for logic programs with variables was established (in [14, Corollary 7.7]). For the present version of QEL the correspondence can be described as follows.

Proposition 10 ([15]). *Let Γ be a universal theory in $\mathcal{L} = \langle C, P \rangle$. Let $\langle U, T, T \rangle$ be a total QHT^s model of Γ. Then $\langle U, T, T \rangle$ is an equilibrium model of Γ iff $\langle T, T \rangle$ is a propositional equilibrium model of $gr_U(\Gamma)$.*

By convention, when $\mathcal{P}$ is a logic program with variables we consider the models and equilibrium models of its universal closure expressed as a set of logical formulas. So from Proposition 10 we obtain:

Corollary 11. *Let $\mathcal{P}$ be a logic program. A total QHT^s model $\langle U, T, T \rangle$ of $\mathcal{P}$ is an equilibrium model of $\mathcal{P}$ iff it is a generalised open answer set of $\mathcal{P}$.*

Proof. It is well-known that for propositional programs equilibrium models coincide with answer sets [12]. Using Proposition 10 and Definition 4 for generalised open answer sets, the result follows. □

6 Relation between Hybrid KBs and QEL

In this section we show how equilibrium models for hybrid knowledge bases[12] relate to the NM models defined earlier and that QEL captures the various approaches to the semantics of hybrid KBs presented in Section 4.

Given a hybrid KB $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ we call $\mathcal{T} \cup \mathcal{P} \cup st(\mathcal{T})$ the *stable closure* of $\mathcal{K}$, where $st(\mathcal{T}) = \{\forall x(p(x) \vee \neg p(x)) : p \in \mathcal{L}_\mathcal{T}\}$.[13] From now on, unless otherwise clear from context, the symbol '$\models$' denotes the truth relation for QHT^s. Given a ground program $\mathcal{P}$ and an $\mathcal{L}$-structure $\mathcal{M} = \langle U, H, T \rangle$, the *projection* $\Pi(\mathcal{P}, \mathcal{M})$ is understood to be defined relative to the component T of $\mathcal{M}$.

Lemma 12. *Let $\mathcal{M} = \langle U, H, T \rangle$ be a QHT^s-model of $\mathcal{T} \cup st(\mathcal{T})$. Then $\mathcal{M} \models \mathcal{P}$ iff $\mathcal{M}|_{\mathcal{L}_\mathcal{P}} \models \Pi(gr_U(\mathcal{P}), \mathcal{M})$.*

Proof. By the hypothesis $\mathcal{M} \models \{\forall x(p(x) \vee \neg p(x)) : p \in \mathcal{L}_\mathcal{T}\}$. It follows that $H|_{\mathcal{L}_\mathcal{T}} = T|_{\mathcal{L}_\mathcal{T}}$. Consider any $r \in \mathcal{P}$, such that $r^\Pi \neq \emptyset$. Then there are four cases to consider. (i) r has the form $\alpha \to \beta \vee p(t)$, $p(t) \in \mathcal{L}_\mathcal{T}$ and $p(\sigma(t)) \notin T$, so $\mathcal{M} \models \neg p(t)$. W.l.o.g. assume that $\alpha, \beta \in \mathcal{L}_\mathcal{P}$, so $r^\Pi = \alpha \to \beta$ and

$$\mathcal{M} \models r \Leftrightarrow \mathcal{M} \models r^\Pi \Leftrightarrow \mathcal{M}|_{\mathcal{L}_\mathcal{P}} \models r^\Pi \qquad (3)$$

by the semantics for QHT^s and Theorem 7. (ii) r has the form $\alpha \to \beta \vee \neg p(t)$, where $p(\sigma(t)) \in T$; so $p(\sigma(t)) \in H$ and $\mathcal{M} \models p(t)$. Again it is easy to see that (3) holds. Case (iii): r has the form $\alpha \wedge p(t) \to \beta$ and $p(\sigma(t)) \in H, T$, so $\mathcal{M} \models p(t)$. Case (iv): r has the form $\alpha \wedge \neg p(t) \to \beta$ and $\mathcal{M} \models \neg p(t)$. Clearly for these two cases (3) holds as well. It follows that if $\mathcal{M} \models \mathcal{P}$ then $\mathcal{M}|_{\mathcal{L}_\mathcal{P}} \models \Pi(gr_U(\mathcal{P}), \mathcal{M})$.

To check the converse condition we need now only examine the cases where $r^\Pi = \emptyset$. Suppose this arises because $p(\sigma(t)) \in H, T$, so $\mathcal{M} \models p(t)$. Now, if $p(t)$ is in the head of r, clearly $\mathcal{M} \models r$. Similarly if $\neg p(t)$ is in the body of r, by the semantics $\mathcal{M} \models r$. The cases where $p(\sigma(t)) \notin T$ are analogous and left to the reader. Consequently if $\mathcal{M}|_{\mathcal{L}_\mathcal{P}} \models \Pi(gr_U(\mathcal{P}), \mathcal{M})$, then $\mathcal{M} \models \mathcal{P}$. □

We now state the relation between equilibrium models and NM-models.

Theorem 13. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be a hybrid knowledge base. Let $\mathcal{M} = \langle U, T, T \rangle$ be a total here-and-there model of the stable closure of $\mathcal{K}$. Then $\mathcal{M}$ is an equilibrium model if and only if it is an NM-model of $\mathcal{K}$.*

Proof. Assume the hypothesis and suppose that $\mathcal{M}$ is in equilibrium. Since $\mathcal{T}$ contains only predicates from $\mathcal{L}_\mathcal{T}$ and $\mathcal{M} \models \mathcal{T} \cup st(\mathcal{T})$, evidently

$$\mathcal{M}|_{\mathcal{L}_\mathcal{T}} \models \mathcal{T} \cup st(\mathcal{T}) \qquad (4)$$

and so in particular $(U, \mathcal{M}|_{\mathcal{L}_\mathcal{T}})$ is a model of $\mathcal{T}$. By Lemma 12,

$$\mathcal{M} \models \mathcal{P} \Leftrightarrow \mathcal{M}|_{\mathcal{L}_\mathcal{P}} \models \Pi(gr_U(\mathcal{P}), \mathcal{M}). \qquad (5)$$

We claim (i) that $\mathcal{M}|_{\mathcal{L}_\mathcal{P}}$ is an equilibrium model of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$. If not, there is a model $\mathcal{M}' = \langle H', T' \rangle$ with $H' \subset T' = T|_{\mathcal{L}_\mathcal{P}}$ and $\mathcal{M}' \models \Pi(gr_U(\mathcal{P}), \mathcal{M})$. Lift $(U, \mathcal{M}')$ to a (first order) $\mathcal{L}$-structure $\mathcal{N}$ by interpreting each $p \in \mathcal{L}_\mathcal{T}$ according to $\mathcal{M}$. So $\mathcal{N}|_{\mathcal{L}_\mathcal{T}} = \mathcal{M}|_{\mathcal{L}_\mathcal{T}}$ and by (4) clearly $\mathcal{N} \models \mathcal{T} \cup st(\mathcal{T})$. Moreover, by Lemma 12 $\mathcal{N} \models \mathcal{P}$ and by assumption $\mathcal{N} \lhd \mathcal{M}$, contradicting the assumption that $\mathcal{M}$ is an equilibrium model of $\mathcal{T} \cup st(\mathcal{T}) \cup \mathcal{P}$. This establishes (i). Lastly we note that since $\langle T|_{\mathcal{L}_\mathcal{P}}, T|_{\mathcal{L}_\mathcal{P}} \rangle$ is an equilibrium model of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$, $\mathcal{M}|_{\mathcal{L}_\mathcal{P}}$ is a generalised open answer set of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ by Corollary 11, so that $\mathcal{M} = \langle U, T, T \rangle$ is an NM-model of $\mathcal{K}$.

For the converse direction, assume the hypothesis but suppose that $\mathcal{M}$ is not in equilibrium. Then there is a model $\mathcal{M}' = \langle U, H, T \rangle$ of $\mathcal{T} \cup st(\mathcal{T}) \cup \mathcal{P}$, with $H \subset T$. Since $\mathcal{M}' \models \mathcal{P}$ we can apply Lemma 12 to conclude that $\mathcal{M}'|_{\mathcal{L}_\mathcal{P}} \models \Pi(gr_U(\mathcal{P}), \mathcal{M}')$. But clearly

$$\Pi(gr_U(\mathcal{P}), \mathcal{M}') = \Pi(gr_U(\mathcal{P}), \mathcal{M}).$$

However, since evidently $\mathcal{M}'|_{\mathcal{L}_\mathcal{T}} = \mathcal{M}|_{\mathcal{L}_\mathcal{T}}$, thus $\mathcal{M}'|_{\mathcal{L}_\mathcal{P}} \lhd \mathcal{M}|_{\mathcal{L}_\mathcal{P}}$, so this shows that $\mathcal{M}|_{\mathcal{L}_\mathcal{P}}$ is not an equilibrium model of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ and therefore $T|_{\mathcal{L}_\mathcal{P}}$ is not an answer set of $\Pi(gr_U(\mathcal{P}), \mathcal{M})$ and $\mathcal{M}$ is not an NM- model of $\mathcal{K}$. □

This establishes the expected connections with the various special types of hybrid KBs discussed earlier.

[12]Again, we restrict ourselves to the equality-free case here.

[13]Evidently $\mathcal{T}$ becomes *stable* in $\mathcal{K}$ in the sense that $\forall \varphi \in \mathcal{T}$, $st(\mathcal{T}) \models \neg\neg\varphi \to \varphi$. The terminology is drawn from intuitionistic logic and mathematics.

Corollary 14. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be a g-hybrid (resp. an r$^+$-hybrid) knowledge base. Let $\mathcal{M} = \langle U, T, T \rangle$ be a total here-and-there model of the stable closure of $\mathcal{K}$. Then $\mathcal{M}$ is an equilibrium model if and only if it is an NM-model (resp. r$^+$-NM-model) of $\mathcal{K}$.*

Proof. As noted, the semantics of [9] for g-hybrid KBs corresponds directly to the NM-models defined here. For the case of an r$^+$-hybrid KB we use the safety condition and apply Proposition 4. □

Corollary 15. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{P})$ be a r-hybrid or an r_w-hybrid knowledge base. Let $\mathcal{M} = \langle U, T, T \rangle$ be an Herbrand model of the stable closure of $\mathcal{K}$. Then $\mathcal{M}$ is an equilibrium model in the sense of [14] if and only if it is an r-NM-model of $\mathcal{K}$.*

Proof. As noted earlier, the version of QEL presented in [14] uses UNA-models. We apply the relationship to answer sets established in [14] together with the definition of the r-NM-models. □

Alternatively we can express Corollary 15 using the version of QEL presented here and assuming SNA-models.

7 Discussion

We have seen that quantified equilibrium logic captures three of the main approaches to integrating classical, first-order or DL knowledge bases with nonmonotonic rules under the answer set semantics. However, QEL has a quite distinct flavour from those of r-hybrid, r$^+$-hybrid and g-hybrid KBs. Each of these hybrid approaches has a semantics composed of two different components: a classical model on the one hand and an answer set on the other. Integration is achieved by the fact that the classical model serves as a pre-processing tool for the rule base. The style of QEL is different. There is one semantics and one kind of model that covers both types of knowledge. There is no need for any pre-processing of the rule base. In this sense the integration is more far-reaching. The only distinction we make is that for that part of the knowledge base considered to be classical and monotonic we add a stability condition to obtain the intended interpretation.

There are other features of the approach using QEL that are worth highlighting. First, it is based on a simple, minimal model semantics in a known non-classical logic, actually a quantified version of Gödel's 3-valued logic. No reducts are involved and consequently the equilibrium construction applies directly to arbitrary first-order theories. The rule part $\mathcal{P}$ of a knowledge base might therefore comprise say a nested logic program, where the heads and bodies of rules may be arbitrary boolean formulas, or perhaps rules permitting nestings of the implication connective. In the propositional case, answer sets have been defined for such general formulas, but more work would be needed to provide integration in a hybrid KB setting. Evidently QEL in the general case is undecidable, so for extensions of the rule language syntax for practical applications one may wish to study restrictions analogous to safety or guardedness.

A second feature of the use of the logic QHT^s is that it can be applied to characterise properties such as the strong equivalence of programs and theories [15]. While strong equivalence and related concepts have been much studied recently in ASP, their characterisation in the case of hybrid knowledge systems remains uncharted territory. The fact that QEL provides a single semantics for hybrid KBs means that a simple concept of strong equivalence should be applicable to such KBs and characterisable using the underlying logic, QHT^s. We view this as a promising topic for future study.

It is interesting to note here that meaning-preserving relations among ontologies have recently become a topic of interest in the DL community, where logical concepts such as that of conservative extension are currently being studied and applied [6]. A unified, logical approach to hybrid KBs should lend itself to well to the application of such concepts.

8 Related Work and Conclusions

We have provided a general notion of hybrid knowledge base, combining first-order theories with nonmonotonic rules, with the aim of comparing and contrasting some of the different variants of hybrid KBs found in the literature. We presented a version of quantified equilibrium logic, QEL, without the unique names assumption, and showed how for a hybrid knowledge base $\mathcal{K}$ there is a natural correspondence between the nonmonotonic models of $\mathcal{K}$ and the equilibrium models of what we call the *stable closure* of $\mathcal{K}$. This yields a way to capture in QEL the semantics of the g-hybrid KBs of Heymans *et al* [9] and the r-hybrid KBs of Rosati [17], where the latter is defined without the UNA but for safe programs. Similarly, the version of QEL with UNA captures the semantics of r-hybrid KBs as defined in [16, 18]. It is important to note that the aim of this paper was not that of providing new kinds of safety conditions or decidability results; these issues are ably dealt with in the literature reviewed here. Rather our objective has been to show how classical and nonmonotonic theories might be unified under a single semantical model. In part, as [9] show with their reduction of DL knowledge bases to answer set programs, this can also be achieved (at some cost of translation) in other approaches. What distinguishes QEL is the fact that it is based on a standard, nonclassical logic, QHT^s, which can therefore provide a unified logical foundation for such extensions of (open) ASP.

There are several other approaches to combining languages for Ontologies with nonmonotonic rules, most importantly dl-programs [5] and its generalisation, HEX-programs [4]. The orthogonal view of integration by entailment instead of integration on the basis of single models [3] taken by this approach can probably not be captured by a simple embedding like the hybrid approaches. Also, the investigation of connections to other combinations of nonmonotonic logics and rules languages with classical logics is still on our agenda. For instance, combinations with Defeasible Logic [19], variants of Autoepistemic Logic [2], or the logic of minimal knowledge and negation as failure (MKNF) [11] have been recently proposed in the literature.

In future work we hope to consider further aspects of applying QEL to the domain of hybrid knowledge systems. Extending the language with functions symbols and with strong negation is a routine task, since QEL includes these items already. We also plan to consider in the future how QEL can be used to define a catalogue of logical relations between hybrid KBs.

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